PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 3, March 1986

SMALL COMMUTATORS AND INVARIANT SUBSPACES

HUAXIN LIN

ABSTRACT. Let A be an operator on a Banach space, and let T be a nonzero compact (polynomial compact) operator. We prove that if $TA - AT$ is "small", then A has a nontrivial (hyper)invariant subspace.

Let A be a nonscalar operator in $\mathcal{B}(E)$, where E is a Banach space, let T be a nonzero compact operator and let $C_T = TA - AT$. V. I. Lomonosov, J. Daughtry, and H. W. Kim, Carl Pearcy and A. L. Shields [**1–3**] proved that when C_T is very "small" (namely, has rank one or zero), A has a nontrivial hyperinvariant subspace. We show here that for some other classes of operators T, when C_T is very "small", A has a nontrivial hyperinvariant subspace.

We need the following lemma.

LEMMA 1. *Let $A \in \mathcal{B}(E)$. If $\lambda \in \sigma(A)$, then $P(A)$ is compact, where $P(t)$ is a polynomial, and if $P(\lambda) \neq 0$, then $\lambda \in \sigma_p(A)$.*

PROOF. Clearly λ is on the boundary of $\sigma(A)$, hence there is $\{f_n\} \in E$, $\|f_n\| = 1$, with $\|Af_n - f_n\| \to 0$. Hence $\|P(A)f_n - P(\lambda)f_n\| \to 0$. Since $P(A)$ is compact, we may assume that $\|P(A)f_n - f_0\| \to 0$ for some $f_0 \in E$. Let $f_* = f_0/P(\lambda)$. Then

$$\|P(\lambda)f_n - P(\lambda)f_*\| \leq \|P(\lambda)f_n - P(A)f_n\| + \|P(A)f_n - P(\lambda)f_*\| \to 0.$$

Hence

$$\|P(\lambda)f_n - P(\lambda)f_*\| \to 0.$$

Since $P(\lambda) \neq 0$, we have $\|f_n - f_n\| \to 0$ and $\|f_*\| = 1$. Hence

$$\|Af_* - f_*\| \leq \|Af_* - Af_*\| + \|Af_n - f_n\| + \|f_n - f_*\| \to 0.$$

That is $Af_* = \lambda f_*$. So $\lambda \in \sigma_p(A)$.

THEOREM 1. *Let $A \in \mathcal{B}(E)$ be nonscalar. If there is $T \in \mathcal{B}(E)$ with $P(T)$ compact (for some polynomial), but $\sigma(P(T)) \neq \{0\}$ such that $C_T = TA - AT$ has rank one, then A has a nontrivial hyperinvariant subspace.*

PROOF. There is $\alpha \neq 0$, $\alpha \in \sigma_p(P(T))$. Hence there is $\mu \in \sigma_p(T)$ such that $P(\mu) = \alpha$. Moreover, $\ker(T - \mu) \subset \ker(P(T) - P(\mu))$. Hence $0 < \dim\ker(T - \mu) < \infty$.

Let $\operatorname{ran}(T - \mu)^0$ be the annihilator of $\operatorname{ran}(T - \mu)$ in E^*. Then $\operatorname{ran}(T - \mu)^0 = \ker(T^* - \overline{\mu})$. By Lemma 1, $\overline{\mu} \in \sigma_p(T^*)$. Moreover, $\ker(T^* - \overline{\mu}) \subset \ker(\overline{P}(T^*) - \overline{P}(\overline{\mu}))$. Since $\overline{P}(\overline{\mu}) = 0$, we have

$$0 < \dim\ker(T^* - \overline{\mu}) < \infty.$$

Received by the editors March 5, 1984 and, in revised form, January 18, 1985.
1980 *Mathematics Subject Classification*. Primary 47A15; Secondary 47B47.

Since $\dim E/\operatorname{cl}\operatorname{ran}(T-\mu) = \dim(\operatorname{ran}(T-\mu)^0)$,

$$0 < \dim E/\operatorname{cl}\operatorname{ran}(T-\mu).$$

But $(T-\mu)A - A(T-\mu) = C_T$, so it follows from Theorem 1 in [**3**] that A has a nontrivial hyperinvariant subspace.

PROPOSITION 1. *Let $A \neq \lambda$ be in $\mathcal{B}(E)$. If T is a nonscalar algebraic operator in $\mathcal{B}(E)$ such that $C_T = TA - AT$ has rank one or less, then A has a nontrivial invariant subspace.*

PROOF. Let $\lambda \in \sigma_p(T)$ and $(T-\lambda)A - A(T-\lambda) = C_T$. If $\ker(T-\lambda) \in \operatorname{Lat} A$, there is $f_0 \in \ker(T-\lambda)$, $f_0 \neq 0$, such that $C_T f_0 = (T-\lambda)Af_0 \neq 0$. This implies that $\operatorname{ran}(C_T) \subset \operatorname{ran}(T-\lambda)$. Then $A(T-\lambda) = (T-\lambda)A - C_T$. Hence, $\operatorname{cl}\operatorname{ran}(T-\lambda) \in \operatorname{Lat} A$. Since A is nonscalar algebraic, $\operatorname{cl}\operatorname{ran}(T-\lambda)$ is a nontrivial subspace.

Let $A \neq \lambda$ be in $\mathcal{B}(H)$ and let T be a nonzero compact operator. If $C_T = TA - AT$ commutes with T, then the spectrum of C_T becomes $\{0\}$ [**4**]. Under this condition, do we have the conclusion that A has a nontrivial hyperinvariant subspace?

The following proposition shows that this condition is too weak to give a result.

PROPOSITION 2. *If $\dim E > 2$, then for every $A \in \mathcal{B}(E)$ there is a nonzero compact operator T such that $C_T = TA - AT$ commutes with T.*

PROOF. If $h \in E$, then since $\dim E > 2$, there is $g \in E$, $g \in \operatorname{span}\{h, Ah\}$. Hence we have a functional F, with $\ker F \supset \operatorname{span}\{h, Ah\}$, $\|F(g)\| = 1$. Let $Tf = F(f)h$ for all $f \in E$. Then T is compact and $T^2 = 0$.

Now $TC_T - C_T T = T(TA - AT) - (TA - AT)T = -2TAT$. But

$$TATf = F(f)TAh = F(f)F(Ah)h = 0.$$

Hence $TC_T - C_T T = 0$.

The point is that when T is very "small", C_T must be very "small". So we need T "larger" than C_T.

THEOREM 2. *Let $A \neq \lambda$ be in $\mathcal{B}(E)$. If there is a nonzero compact T such that $C_T = TA - AT$ commutes with T and*

$$\operatorname*{Lim}_{k \to \operatorname{rank} C_T} \|T^k\|^{1/k} \neq 0,$$

then A has a nontrivial hyperinvariant subspace.

PROOF. If $\operatorname{Lim}_{k\to\infty} \|T^k\|^{1/k} \neq 0$, i.e., $r(T) > 0$, there is $\alpha \neq 0$, $\alpha \in \sigma_p(T)$. Hence $\bar{\alpha} \in \sigma_p(T^*)$. It is well known that

$$\dim \bigvee_{n=1}^{\infty} \ker(T^* - \alpha)^n < \infty.$$

Hence there is an integer $k > 0$ such that $\ker(T^* - \bar{\alpha})^{k-1} = \ker(T^* - \bar{\alpha})^k$ since $\ker(T^* - \bar{\alpha})^n \subset \ker(T^* - \bar{\alpha})^{n+1}$ for all n. In other words,

$$\operatorname{cl}[\operatorname{ran}(T-\alpha)^{k-1}] = \operatorname{cl}[\operatorname{ran}(T-\alpha)^k].$$

Since $(T-\alpha)A - A(T-\alpha) = C_T$ and $T - \alpha$ commutes with C_T, by induction we have

$$(T-\alpha)^k A - A(T-\alpha)^k = k(T-\alpha)^{k-1} C_T.$$

Hence

$$A(T-\alpha)^k = (T-\alpha)^k A - k(T-\alpha)^{k-1}C_T.$$

This implies $A(T-\alpha)^k f \in \operatorname{ran}[(T-\alpha)^k]$ for all $f \in H$. Hence

$$\operatorname{cl}\operatorname{ran}[(T-\alpha)^k] \in \operatorname{Lat} A.$$

Hence $\ker[(T^*-\bar{\alpha})^k] \in \operatorname{Lat} A^*$. But $0 < \dim[(T^*-\bar{\alpha})^k] < \infty$. Hence $A^*|_{\ker(T^*-\bar{\alpha})^k}$ has an eigenvalue $\overline{\lambda}$. But $A^* \neq \overline{\lambda}$. Hence $\{0\} \neq \ker(A^*-\overline{\lambda}) \neq E$. This implies that $\operatorname{cl}[\operatorname{ran}(A-\lambda)]$ is a nontrivial hyperinvariant subspace of A.

If $\operatorname{Lim}_{k\to\infty}\|T^*\|^{1/k} = 0$, the condition implies that $\dim(\operatorname{ran} C_T) = k$, with $T^k \neq 0$ for some $k > 0$. Since $TC_T = C_T T$, $\operatorname{cl}[\operatorname{ran} C_T] \in \operatorname{Lat} T$. But $\dim(\operatorname{ran} C_T) = k$, hence there is $\lambda \in \sigma_p(T_{\operatorname{ran} C_T}) \subset \sigma_p(T)$. Hence $\lambda = 0$, i.e., there is $f_0 \neq 0$, $f_0 \in \operatorname{ran} C_T$ with $Tf_0 = 0$. Hence $\dim[\operatorname{ran}(TC_T)] \leq k-1$.

By induction, since $T(T^i C_T) = (T^i C_T)T$, we have

$$\dim[\operatorname{ran}(T^{k-1}C_T)] \leq 1.$$

On the other hand, $T^k A - AT^k = kT^{k-1}C_T$, $T^k \neq 0$ is compact. It follows from [**3**] that A has a nontrivial hyperinvariant subspace.

THEOREM 3. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(E)$, where E is a Banach space. Then* $\operatorname{Lat}\mathcal{A} = \{0, E\}$ *if and only if for all subalgebras $\mathcal{A}'$ of $\mathcal{B}(E)$ with $\mathcal{A}' \supset \mathcal{A}$,* $\operatorname{rad}\mathcal{A}' \cap \mathcal{C}(E) = 0$.

PROOF. If $\operatorname{Lat}\mathcal{A} = \{0, E\}$, then $\operatorname{Lat}\mathcal{A}' = \{0, E\}$ for every $\mathcal{A}' \supset \mathcal{A}$. If $\operatorname{rad}\mathcal{A}' \cap \mathcal{C}(E) \neq 0$ for some $\mathcal{A}' \supset \mathcal{A}$, let $A \in \operatorname{rad}\mathcal{A}' \cap \mathcal{C}(E)$, $A \neq 0$. Then for every $B \in \mathcal{A}'$, $1 - BA$ is invertible. By Lomonosov's lemma, there is a $B \in \mathcal{A}'$ such that $1 \in \sigma_p(BA)$. Thus we have a contradiction.

Conversely, suppose $E_0 \in \operatorname{Lat}\mathcal{A}$, $0 \neq E_0 \neq E$. There is $F \in E^*$ such that $\|F\| = 1$, $\ker F \supset E_0$. Let $f_0 \in E_0$, $f_0 \neq 0$, and define $Af = F(f)f_0$ for every $f \in E$. Then A is a compact operator and $E_0 \in \operatorname{Lat} A$. Let $\mathcal{A}'$ be the algebra generated by $\mathcal{A}$ and A. Let $I = \{B\colon B|_E = 0,\ B \in \mathcal{A}'\}$. Clearly I is a two sided ideal of $\mathcal{A}'$ and $A \in I$. Let J be a maximal right ideal of $\mathcal{A}'$. Consider the homomorphism $\pi\colon \mathcal{A}' \to \mathcal{A}'/I = \overline{\mathcal{A}}$. Then $\pi(J)$ is a right ideal of $\overline{\mathcal{A}}$. Let J' be $\pi^{-1}(J)$. Then $J' \supset J \cup I$ and J' is a right ideal. Hence $I \subset J$, which implies $A \subset J$, or $J' = \mathcal{A}'$. If $J' = \mathcal{A}'$, then $\pi(1)$ is the indentity of $\overline{\mathcal{A}}$ which is in $\pi(J)$. Let $i \in J$ such that $\pi(i) = \pi(1)$. Then $iA \in J$. But $iA = A$, since $\pi(i)$ is the identity of $\overline{\mathcal{A}}$. This implies $A \in J$. Hence every maximal right ideal J contains A. So $A \in \operatorname{rad}(\mathcal{A}')$, and $\operatorname{rad}(\mathcal{A}') \cap \mathcal{C}(E) \neq 0$. This proves the theorem.

COROLLARY. *Let $A \neq \lambda$ be in $\mathcal{B}(E)$, where E is a Banach space, and let T be a nonzero compact operator. Let $\mathcal{A}$ be generated by $1, A$ and T (or by $\{A\}'$ and T). Then if $C_T = TA - AT$ is so "small" that $C_T \in \operatorname{rad}\mathcal{A}$, then A has a nontrivial invariant (hyperinvariant) subspace.*

PROOF. This is obvious by Theorem 3 and Lomonosov's lemma [**1**].

Let A be quasinilpotent. Let $\mathcal{A}$ be the subalgebra generated by A and 1, or by $\{A\}'$. Then $\operatorname{rad}\mathcal{A} \neq \{0\}$.

Question. Is there any nonzero compact operator T such that the subalgebra $\mathcal{R}$ generated by $\mathcal{A}$ and T has $\operatorname{rad}(\mathcal{R}) \neq 0$?

REFERENCES

1. V. I. Lomonosov, *Invariant subspaces for the family of operators which commutes with a completely continuous operator*, Functional Anal. Appl. **7** (1973), 213–214.
2. J. Daughtry, *An invariant subspace theorem*, Proc. Amer. Math. Soc. **49** (1975), 267–268.
3. H. W. Kim, Carl Pearcy and A. L. Shields, *Rank one commutator and hyperinvariant subspaces*, Michigan Math. J. **22** (1976), 193–194.
4. P. R. Halmos, *A Hilbert space problem book*, Problem 184, Van Nostrand-Reinhold, Princeton, N. J., 1967.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907