PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 3, March 1986

SECOND DERIVATIVE L^p-ESTIMATES FOR ELLIPTIC EQUATIONS OF NONDIVERGENT TYPE

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ABSTRACT. We obtain an a priori estimate of second derivatives in L^p, for some $p > 0$, for solutions of nondivergent, uniformly elliptic P.D.E.'s of second order.

1. Introduction. Let $\Omega \subseteq \mathbf{R}^n$ be a smooth, bounded domain, and let $a \equiv (a_{ij})$ be a measurable, symmetric $n \times n$ matrix-valued function on Ω which satisfies

$$\lambda I \leq a(x) \leq \lambda^{-1} I, \quad \text{for some constant} \tag{1.1}$$

$\lambda \in (0,1)$ and a.e. $x \in \Omega$, in the sense of nonnegative definiteness. Set

$$L_a \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

and let $u \in C^{1,1}(\Omega)$ be the solution of the following problem:

$$\begin{cases} L_a u = -f & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $f \in L^n(\Omega)$.

The main aim of this note is to prove the following

THEOREM. *There is a positive constant $p = p(n, \lambda)$ such that*

$$\|D^2 u\|_{L^p(\Omega)} \leq c(n, \lambda, p, \Omega) \|f\|_{L^n(\Omega)}, \tag{1.3}$$

for any $u \in C^{1,1}(\Omega)$; $u|_{\partial\Omega} = 0$. Here $f = L_a u$ and $D^2 u$ is the Hessian matrix of u.

It should be noted that for every p, $1 \leq p < +\infty$, and $n \geq 3$ there is an operator L_a which satisfies (1.1), but an a priori estimate of the form

$$\|D^2 u\|_{L^p(\Omega)} \leq c(n, \lambda, p, \Omega) \|f\|_{L^p(\Omega)}, \tag{1.4}$$

where u satisfies (1.2), is not true. See, for example, [**4** and **6**].

As far as inequality (1.3) is concerned, we can assume that a is a smooth matrix-valued function on Ω. In such a case we will study the behavior of nonnegative solutions, v, to the adjoint equation

$$L_a^* v \equiv \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} [a_{ij}(x) v] = 0, \quad \text{in } \Omega. \tag{1.5}$$

Received by the editors March 10, 1985.
1980 *Mathematics Subject Classification*. Primary 35J15; Secondary 35B45.
[1]Research partially supported by the Alfred P. Sloan Foundation.

It is shown by Fabes and Stroock [**3**] that

$$\left[\frac{1}{|B|}\int_B v(y)^{n/(n-1)}\,dy\right]^{(n-1)/n} \le c(n,\lambda)\frac{1}{|B|}\int_B v(y)\,dy \tag{1.6}$$

for all balls B whose concentric double is contained in Ω.

(1.6) combines with a simple argument (see §2 below) to conclude the following inequality for the solution u of (1.2):

$$\|D^2u\|_{L^p(\Omega')} \le c(n,p,\lambda,\Omega',\Omega)\|f\|_{L^n(\Omega)} \tag{1.7}$$

for all $\Omega' \Subset \Omega$ and $p = p(n,\lambda) > 0$.

In order to prove (1.3) we will need a version of (1.6) near the boundary, $\partial\Omega$, of Ω. For this purpose we normalize Ω (by a proper scaling) so that the absolute value of the curvatures of $\partial\Omega$ is bounded by one. Then we have the following

LEMMA. *Let $v \ge 0$ be a solution of the adjoint equation in the set $\{x \in \Omega : \operatorname{dis}(x,\partial\Omega) < 1\}$, and $v|_{\partial\Omega} = 0$. Then*

$$\left[\frac{1}{|B|}\int_{B\cap\Omega} v(y)^{n/(n-1)}\,dy\right]^{(n-1)/n} \le c(n,\lambda)\frac{1}{|B|}\int_{B\cap\Omega} v(y)\,dy \tag{1.8}$$

for all balls $B = B_r(x)$ such that $\operatorname{dis}(x,\partial\Omega) \le r/3$ *and* $r < 1/3$.

The proof of the lemma, which is similar to that in [**3**], was suggested to me by Professor E. Fabes. The author wishes to express his gratitude to Professor Fabes for several interesting discussions.

2. Proof of the Theorem.

First we note that, by [**1**], estimates (1.6) and (1.8) will imply that for any measurable subset $E \subset \Omega$,

$$\int_E G_a(\underline{o},y)\,dy \ge c(\lambda,n,\Omega)|E|^m, \tag{2.1}$$

for some $m = m(\lambda,n) > 0$, where $G_a(x,y)$ is Green's function of the operator L_a on Ω, and $\underline{o} \in \Omega$ is a fixed point (but arbitrarily chosen). For the details of the proof we refer to [**3**]; see also [**2**, Theorem 1].

Now we claim that (2.1) *implies* (1.3).

PROOF OF THE CLAIM. Let $u \in C^{1,1}(\Omega)$ solve (1.2). Without loss of generality we assume that $\|f\|_{L^n(\Omega)} = 1$, and we want to show that

$$\|D^2u\|_{L^p(\Omega)} \le c(n,\lambda,p,\Omega), \quad \text{for some } p = p(n,\lambda) > 0, \tag{2.2}$$

where D^2u is the Hessian matrix of the function u.

By $L_a u \equiv \operatorname{Tr}(A\cdot D^2u) = \sum_{i=1}^k \alpha_i\lambda_i - \sum_{j=k+1}^n \alpha_j\lambda_j$, where $\lambda_1, \lambda_2, \ldots, \lambda_k, -\lambda_{k+1}, \ldots, -\lambda_n$ are eigenvalues of the matrix (D^2u) with λ_l's nonnegative, $k = k(x)$, and α_i, $i = 1,\ldots,n$, are functions satisfying $\lambda \le \alpha_i \le \lambda^{-1}$ for $i = 1,2,\ldots,n$. Thus $L_a u = \alpha P - \beta N$, where $\lambda \le \alpha \le \lambda^{-1}$, $\lambda \le \beta \le \lambda^{-1}$, and $P = \sum_{i=1}^k \lambda_i$, $N = \sum_{j=k+1}^n \lambda_j$.

Now we introduce a new operator, which will depend on u, as follows: If $x \in \Omega$, and $Q(x) \in O(n)$ (i.e., orthogonal matrices),

$$Q(x)^{\mathrm{t}}(D^2u(x))Q(x) = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_k, -\lambda_{k+1}, \ldots, -\lambda_n\},$$

then set

$$a^*(x) = Q(x)^t \mathrm{diag}\{\tfrac{1}{2}\lambda, \ldots, \tfrac{1}{2}\lambda, +2\lambda^{-1}, \ldots, +2\lambda^{-1}\} Q(x)$$

so that

$$\tfrac{1}{2}\lambda I \le a^*(x) \le 2\lambda^{-1} I, \quad \text{for a.e. } x \in \Omega,$$

and a^* will be measurable since $u \in C^{1,1}(\Omega)$. Moreover,

$$L_{a^*} u \equiv L_a u - \gamma |D^2 u| = -f - \gamma |D^2 u| \tag{2.3}$$

for some measurable function $\gamma \in L^\infty(\Omega)$, with $\gamma \ge \lambda/2$.

Now let a_ε^* be a smoothing of a^* so that $\frac{1}{2}\lambda I \le a_\varepsilon^*(x) \le 2\lambda^{-1} I$, for $x \in \Omega$. Then

$$L a_{\varepsilon^*} u(x) = -f(x) - \gamma(x)|D^2 u|(x) + \varepsilon_{ij}(x) u_{x_i x_j}(x),$$

where $\varepsilon_{ij}(x) \in L^\infty(\Omega)$ and, as $\varepsilon \to 0^+$, $\varepsilon_{ij}(x) \overset{L^1(\Omega)}{\to} 0$. Now by the maximum principle of Alexandorff, Pucci and Bakelman [**7**], we have

$$\begin{aligned}\int_\Omega G_{a_\varepsilon^*}(\underline{o}, y)|D^2 u(y)|\, dy \le{}& c(\lambda, n, \Omega)\|f\|_{L^n(\Omega)} \\ &+ c(n, \lambda, \Omega)\|\varepsilon_{ij} u_{x_i x_j}\|_{L^n(\Omega)}, \quad \text{for all } \varepsilon > 0.\end{aligned} \tag{2.4}$$

Let $E_t = \{x \in \Omega \colon |D^2 u(x)| \ge t\}$. Then

$$t \int_{E_t} G_{a_\varepsilon^*}(\underline{o}, y)\, dy \le c(n, \lambda, \Omega)\left[1 + \|\varepsilon_{ij} u_{x_i x_j}\|_{L^n(\Omega)}\right]. \tag{2.5}$$

By (2.1) we deduce that

$$|E_t| \le t^{-1/m} c(n, \lambda, \Omega)^{1/m}\left[1 + \|\varepsilon_{ij} u_{x_i x_j}\|_{L^n(\Omega)}\right]^{1/m}. \tag{2.6}$$

This is true for all $\varepsilon > 0$. We let $\varepsilon \to 0^+$ and obtain

$$|E_t| \le c(n, \lambda, \Omega)^{1/m} t^{-1/m}.$$

(1.3) follows if we choose $p = p(n, \lambda) < 1/m$. Q.E.D.

3. Proof of the Lemma. Let v, Ω be as in the lemma, let $x_0 \in \Omega$ with $\mathrm{dis}(x_0, \partial\Omega) < \gamma/3$, $\gamma \in (0, \frac{1}{3})$, and $B_\gamma(x_0) \cap \Omega \equiv \Omega_\gamma(x_0) \subset \Omega_{2\gamma}(x_0) \subset \Omega$. We first have

LEMMA 1.

$$\left[\frac{1}{|\Omega_\gamma|}\int_{\Omega_\gamma} v(y)^{n/(n-1)}\, dy\right]^{(n-1)/n} \le c(n, \lambda)\frac{1}{|\Omega_{3\gamma/2}|}\int_{\Omega_{3\gamma/2}} v(y)\, dy.$$

PROOF. See the proof of Theorem 2.1 in [**3**].

Next we want to show that the measure $v\, dy$ satisfies the "doubling condition" near the boundary.

LEMMA 2. *Let $v, \Omega_\gamma, \Omega_{2\gamma}$ be as above. Then*

$$\int_{\Omega_\gamma} v(y)\, dy \le c(n, \lambda)\int_{\Omega_{\gamma/2}} v(y)\, dy$$

for all $\gamma \in (0, \frac{1}{3})$.

PROOF. By a proper scaling, it will suffice to prove the following statement: If $v \geq 0$ is a solution of an adjoint equation in Ω_4, and $v|_{\partial\Omega} = 0$, where $\partial\Omega$ is smooth and has curvatures bounded by $1/6$, then there is a constant $c = c(n, \lambda)$ such that

$$\int_{\Omega_2} v(y)\,dy \leq c(n,\lambda) \int_{\Omega_1} v(y)\,dy. \tag{3.1}$$

To show (3.1), we smooth the corner of $\partial\Omega_4$ to obtain a new domain $\tilde{\Omega}_4 \subset \Omega_4$, so that $\tilde{\Omega}_4 \supset \Omega_3$ and the curvatures of $\partial\tilde{\Omega}_4$ are bounded by $c(n)$ (a constant depending only on n). Let $G(x, y)$ be Green's function of L_a on $\tilde{\Omega}_4$ (it should be noted that the matrix a here is obtained from (1.1) by a proper scaling in independent variables). Then

$$v(y) = \int_{\partial\tilde{\Omega}_4} \langle a(x) D_x G(x,y), n(x)\rangle v(x)\,ds_x, \qquad \forall y \in \tilde{\Omega}_4, \tag{3.2}$$

where $n(x)$ = the inner unit normal vector of $\partial\tilde{\Omega}_4$ at $x \in \partial\tilde{\Omega}_4$. Set

$$u_i(x) = \int_{\Omega_i} G(x,y)\,dy, \quad \text{for } i = 1, 2. \tag{3.3}$$

Then

$$\int_{\Omega_i} v(y)\,dy = \int_{\partial\tilde{\Omega}_4} \langle a(x) D_x u_i(x), n(x)\rangle v(x)\,ds_x. \tag{3.4}$$

Now we observe that u_1, u_2 have the following properties:
(i) $L_a u_i(x) = -\chi_{\Omega_i}(x)$, for $i = 1, 2$.
(ii) $0 \leq u_i(x) \leq c(n, \lambda)$ in $\tilde{\Omega}_4$, and $u_1|_{\partial\tilde{\Omega}_4} = 0$, for $i = 1, 2$.
(iii) For any $K \Subset \tilde{\Omega}_4$,

$$\inf_{x\in K} u_2(x) \geq \inf_{x\in K} u_1(x) \geq c(n,\lambda,K) > 0.$$

(ii) and (iii) imply that

$$u_1(x) \geq c(n,\lambda,K) u_2(x) \quad \text{on } K. \tag{3.5}$$

By our choice of $\tilde{\Omega}_4$, it is easy to see that there are two positive constants $\delta = \delta(n, \lambda) > 0$ and $c = c(n, \lambda) > 0$ such that
(i) $d(x) \pm cd^2(x) \geq d(x)/2$, for all $x \in \tilde{\Omega}_4$ such that $d(x) \leq \delta$,
(ii) $L_a[d(x) + cd^2(x)] \geq 1$ and $L_a[d(x) - cd^2(x)] \leq -1$ for all $x \in \tilde{\Omega}_4$ with $d(x) \leq \delta$, where $d(x) = \text{dis}(x, \partial\tilde{\Omega}_4)$. See the appendix of [**5**].

By the maximum principle we conclude that there is a positive constant $c = c(n, \lambda)$ such that

$$u_1 \geq (x)c^{-1}(n,\lambda)[d(x) + cd^2(x)], \qquad u_2(x) \leq c(n,\lambda)[d(x) - cd^2(x)] \tag{3.6}$$

for $x \in \tilde{\Omega}_4$ and $\text{dis}(x, \partial\tilde{\Omega}_4) \leq \delta$. (3.6) implies that

$$\langle a(x) D_x u_2(x), n(x)\rangle \leq c(n,\lambda)^2 \langle a(x) D_x u_1(x), n(x)\rangle \qquad \forall x \in \partial\tilde{\Omega}_4. \tag{3.7}$$

By (3.4) we obtain (3.1) and thus complete the proof of Lemma 2. Q.E.D.

Finally, one notices that (1.8) follows from Lemmas 1 and 2 and, hence, completes the proof of the Theorem.

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