PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 3, March 1986

THE MAXIMUM MODULUS PRINCIPLE FOR CR FUNCTIONS

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ABSTRACT. Let M be a CR submanifold of $\mathbf{C}^n$ without extreme points. Then, the modulus of any CR function on M cannot have a strong local maximum at any point of M.

Preliminaries. Let M be a smooth manifold embedded as a locally closed real submanifold of $\mathbf{C}^n$. We denote by $\bar\partial_M$ the tangential Cauchy-Riemann operator on M induced by the Cauchy-Riemann operator $\bar\partial$ on $\mathbf{C}^n$ and with $\mathrm{HT}_p(M)$ the holomorphic tangent space to M at a point $p \in M$.

Let us recall some of the definitions and results of [**4**].

DEFINITION 1. $\bar\partial_M$ *obeys the local maximum modulus principle* on M if given any open connected set U in M and any u differentiable in U such that $\bar\partial_M u = 0$ on U, then u cannot have a (weak) local maximum at any point of U unless u is constant on U.

DEFINITION 2. We call a point $p \in M$ an *extreme point* of M if there exists a local holomorphic coordinate system $z = (z_1, \dots, z_n)$ in a neighborhood U of p such that $z(p) = 0$ and $M \cap U \subset \{z \mid y_1 \geqslant 0\}$. Here we assume that locally near p, M is not contained in any $\mathbf{C}^k$ for $k < n$.

DEFINITION 3. (i) For any $p \in M$ and $X \in \mathrm{HT}_p(M)$ set $Z = X - iY$, where $Y = JX \in \mathrm{HT}_p(M)$ and J is the multiplication with $(-1)^{1/2}$ which defines the complex structure on $\mathbf{R}^{2n}$.

The Levi form at p assigns to Z the normal vector $L_p(Z)$ defined by $L_p(Z) = B_p(X, X) + B_p(Y, Y)$, where B_p is the second fundamental form of M at p.

(ii) We denote $N_p(M)$ as the normal space of M at p.

For any $\xi \in N_p(M)$ the map L_p^ξ defined by $L_p^\xi(Z) = \langle L_p(Z), \xi\rangle$ is called *the Levi form of M at p in the ξ direction*. Here $\langle\ ,\ \rangle$ represents the real inner product in $\mathbf{R}^{2n}$.

We assume that $p = 0$ and $\operatorname{codim}_{\mathbf{R}} M = q$. Then in a neighborhood U of the origin there are smooth real functions $\rho_1, \dots, \rho_q$ such that $d\rho_1 \wedge \cdots \wedge d\rho_{q|0} \neq 0$ and

$$M \cap U = \left\{ z \in U \mid \rho_1(z) = \cdots = \rho_q(z) = 0 \right\}.$$

We may assume that $d\rho_1(0), \dots, d\rho_q(0)$ are orthonormal.

Received by the editors September 27, 1984 and, in revised form, March 13, 1985.
1980 *Mathematics Subject Classification*. Primary 32F25.

If $\xi \in N_0(M)$, then $\xi = \sum_{i=1}^q \xi_i \, d\xi_i(0)$ and if $Z = \sum_{j=1}^n w_j(\partial/\partial z_j) \in \mathrm{HT}_0(M)$, then $\sum_{j=1}^n (\partial\rho_i/\partial z_j)(0) w_j = 0$ for $1 \leqslant i \leqslant q$.

In these conditions

$$L_0^{\xi}(Z) = -4 \sum_{i,j,k} \xi_i \frac{\partial^2 \rho_i}{\partial z_j \partial \bar{z}_k}(0) w_j \bar{w}_k \qquad [\mathbf{4}].$$

In [**4**] the following results are proved.

PROPOSITION 1. *If p is an extreme point of M, then there exists a normal direction $\xi \in N_p(M)$ such that L_p^{ξ} is positive definite.*

PROPOSITION 2. *If for a point $p \in M$ there exists $\xi \in N_p(M)$ such that L_p^{ξ} is strictly positive definite, then p is an extreme point of M.*

THEOREM 1. *If $\bar{\partial}_M$ obeys the local maximum modulus principle on M, then M can contain no extreme point.*

In [**4**], it is conjectured that the converse of Theorem 1 is also true.

Statement of results. A submanifold M of $\mathbf{C}^n$ is called a CR manifold if $\dim_{\mathbf{C}} \mathrm{HT}_p(M)$ is constant on M.

We say that M has CR dimension m if $\dim_{\mathbf{C}} \mathrm{HT}_p(M) = m$, and we denote $\mathrm{CR}\dim(M) = m$.

A totally real submanifold of $\mathbf{C}^n$ is a CR submanifold of $\mathbf{C}^n$ with $\mathrm{CR}\dim(M) = 0$.

A complex valued smooth function f on M for which $\bar{\partial}_M f = 0$ on M is called a CR function on M.

THEOREM 2. *Each point of a totally real submanifold $M \subset \mathbf{C}^n$ is an extreme point of M.*

THEOREM 3. *If M is a CR submanifold of $\mathbf{C}^n$ without extreme points, then for any CR function f on M, $|f|$ cannot have a strong local maximum at any point of M.*

PROOF OF THEOREM 2. We know from [**5**] that there exists a nonnegative function $\varphi \in C^2(\mathbf{C}^n)$ strictly plurisubharmonic in a neighborhood D of M such that

$$M = \{z \in D | \varphi(z) = 0\} = \{z \in D | \operatorname{grad} \varphi = 0\}.$$

Let $p \in M$. We assume that $p = 0$ and in a neighborhood V of p we have $V \cap M = \{z \in V | \rho_1(z) = \cdots = \rho_q(z) = 0\}$ with $d\rho_1 \wedge \cdots \wedge d\rho_{q|0} \neq 0$.

Let $\rho = \varphi + \varepsilon\rho_1 \in C^2(V)$, where $\varepsilon > 0$ is chosen small enough such that the complex Hessian of ρ is strictly positive definite at the origin.

We have also $d\rho(0) = \varepsilon d\rho_1(0) \neq 0$ and we may assume that $(\partial\rho_1/\partial z_1)(0) \neq 0$, so $(\partial\rho/\partial z_1)(0) \neq 0$.

Because $\rho(0) = 0$ in a neighborhood of the origin we have:

$$\rho(z) = 2 \operatorname{Re} \sum_{i=1}^n \frac{\partial\rho}{\partial z_i}(0) z_i + \operatorname{Re}\left(\sum_{i,j=1}^n \frac{\partial^2\rho}{\partial z_i \partial z_j}(0) z_i z_j \right) + \sum_{i,j=1}^n \frac{\partial^2\rho}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j + O\left(|z|^3\right).$$

We make the holomorphic change of coordinates in $\mathbf{C}^n$:

$$z_1' = 2i\sum_{i=1}^{n} \frac{\partial\rho}{\partial z_i}(0)z_i + i\sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial z_j}(0)z_i z_j, \qquad z_i' = z_i \text{ for } 2 \leqslant i \leqslant n.$$

In the new coordinates we have

$$\rho(z') = -y_1' + \sum_{i,j=1}^{n} a_{ij} z_i' \bar{z}_j' + O(|z'|^3)$$

with $\sum_{i,j=1}^{n} a_{ij} z' z_j'$ strictly positive definite.

Define $S = \{z \in V | \rho(z) = 0\}$.

It follows that $S \subset \{z' | y_1' \geqslant 0\}$ in the neighborhood of the origin and because $M \cap V \subset S$ the theorem is proved.

PROOF OF THEOREM 3. We shall use the following lemma [2]:

LEMMA 1. *Let Ω be an open subset of R^N with coordinates $x_1, \ldots, x_N$. Let $F \in C^\infty(\Omega)$ and L be a compact subset of Ω. Suppose that $F(x) < \max_L F$ for each $x \in \Omega - L$. Then for any open set Ω_1 with $L \subset \Omega_1 \subset \Omega$, there exists a point $y \in \Omega_1$ such that the Hessian of F at y is strictly negative definite.*

We denote $d = \dim_R M$, $q = 2n - d$ and $m = \mathrm{CR}\dim M$.

Let us suppose that there exists a CR function f on M such that $|f|$ has a point of strong local maximum. Then there exists a compact set $K \subset M$ such that $\max_K |f| > \max_{\partial K} |f|$. We may assume that K is contained in an open set in R^d which is part of the atlas that defines M. We may assume also that $\max_K \operatorname{Re} f > \max_{\partial K} \operatorname{Re} f$. Let us denote $\operatorname{Re} f = \varphi$.

By Lemma 1 there is $p \in K$ such that $((\partial^2\varphi/\partial t_i \partial t_j)(p))_{1 \leqslant i \leqslant d, 1 \leqslant j \leqslant d}$ is strictly negative definite for any real coordinates $(t_1, \ldots, t_d)$ in a neighborhood of p.

From Theorem 2, we obtain that $m \geqslant 1$. We denote $s = d - 2m$ and $r = m - (d - n)$.

After a complex linear change of coordinates in $\mathbf{C}^n$, M may be represented in the neighborhood of the point p by the equations

$$\begin{aligned} z_j &= x_j + ig^j(x, w), & j &= 1, \ldots, s, \\ z_{j+s} &= h^j(x, w), & j &= 1, \ldots, r, \\ z_{j+s+r} &= w_j, & j &= 1, \ldots, m, \end{aligned}$$

where p corresponds to the origin and $\{g^j\}_{j=1,\ldots,s}$, $\{h^j\}_{j=1,\ldots,r}$ are real and complex valued functions respectively vanishing to second order at the origin [7].

Therefore, if

$$\begin{gathered} \rho_1 = y_1 - g^1(x, z), \ldots, \rho_s = y_s - g^s(x, z), \\ \rho_{s+1} = x_{s+1} - \operatorname{Re} h^1(x, z), \\ \rho_{s+2} = y_{s+1} - \operatorname{Im} h^1(x, z), \ldots, \rho_q = y_{s+r} - \operatorname{Im} h^r(x, z), \end{gathered} \tag{1}$$

where $x = (x_1, \dots, x_s)$ and $z = (z_{s+r+1}, \dots, z_n)$, then in a neighborhood of the origin we have

$$M = \{z | \rho_1(z) = \cdots = \rho_q(z) = 0\},$$

$$d\rho_1 \wedge \cdots \wedge d\rho_{q|0} \neq 0 \quad \text{and} \quad d\rho_1(0), \dots, d\rho_q(0) \text{ are orthonormal.}$$

φ is the real part of a CR function so $(\partial\bar{\partial})_b\varphi = 0$ [3].

If $\tilde{\varphi}$ is a real extension of φ, then in a neighborhood U of p in $\mathbf{C}^n$ we obtain

$$\partial\bar{\partial}\tilde{\varphi} = \sum_{k=1}^{q} a_k\rho_k + \sum_{k=1}^{q} b_k\partial\rho_k + \sum_{k=1}^{q} c_k\bar{\partial}\rho_k + \sum_{k=1}^{q} d_k\partial\bar{\partial}\rho_k$$

with $a_k \in C^\infty_{(1,1)}(U)$, $b_k \in C^\infty_{(0,1)}(U)$, $c_k \in C^\infty_{(1,0)}(U)$ and $d_k \in C^\infty(U)$ or,

$$\frac{\partial^2\tilde{\varphi}}{\partial z_i\partial\bar{z}_j} = \sum_{k=1}^{q} a_{ijk}\rho_k + \sum_{k=1}^{q} b_{kj}\frac{\partial\rho_k}{\partial z_i} + \sum_{k=1}^{q} c_{ki}\frac{\partial\rho_k}{\partial\bar{z}_j} + \sum_{k=1}^{q} d_k\frac{\partial^2\rho_k}{\partial z_i\partial\bar{z}_j}. \tag{2}$$

We choose an extension $\tilde{\varphi}$ of φ which is independent of the variables $y_1, \dots, y_s$, $x_{s+1}, y_{s+1}, \dots, x_{s+r}, y_{s+r}$ which are normal to M over the origin.

Let $Z \in \mathrm{HT}_0(M)$, i.e.

$$Z = \sum_{i=1}^{n} z_i\left(\frac{\partial}{\partial z_i}\right)_0 \quad \text{and} \quad \sum_{i=1}^{n} z_i\frac{\partial\rho_k}{\partial z_i}(0) = 0,$$

$k = 1, \dots, q$.

From (2) we obtain

$$\sum_{i,j=1}^{n} \frac{\partial^2\tilde{\varphi}}{\partial z_i\partial\bar{z}_j}(0)z_i\bar{z}_j = \sum_{i,j=1}^{n}\sum_{k=1}^{q} d_k(0)\frac{\partial^2\rho_k}{\partial z_i\partial\bar{z}_j}(0)z_i\bar{z}_j. \tag{3}$$

But, with the functions ρ_k given by (1), $Z \in \mathrm{HT}_0(M)$ if $z_1 = \cdots = z_{s+r} = 0$. It follows that

$$\sum_{i,j=1}^{n} \frac{\partial^2\tilde{\varphi}}{\partial z_i\partial\bar{z}_j}(0)z_i\bar{z}_j = \sum_{i,j=1}^{m} \frac{\partial^2\varphi}{\partial w_i\partial\bar{w}_j}(0)w_i\bar{w}_j,$$

which is strictly negative definite.

Indeed supposing that

$$\sum_{i,j=1}^{m} \frac{\partial^2\varphi}{\partial w_i\partial\bar{w}_j}(0)w_i\bar{w}_j$$

is not strictly negative definite, by making a complex linear change of coordinates $w_1, \dots, w_m$, in the new coordinates $w'_1, \dots, w'_m$ we may assume

$$\frac{\partial^2\varphi}{\partial w'_1\partial\bar{w}'_1}(0) \geqslant 0.$$

But

$$\frac{\partial^2\varphi}{\partial w'_1\partial\bar{w}'_1}(0) = \frac{1}{4}\left(\frac{\partial^2\varphi}{\partial u'^2_1}(0) + \frac{\partial^2\varphi}{\partial v'^2_1}(0)\right),$$

where $w_1' = u_1' + iv_1'$ and this contradicts the fact that the real Hessian of φ at 0 is strictly negative definite.

Taking the real parts in (3) we obtain

$$\sum_{i,j=1}^{n} \frac{\partial^2\tilde{\varphi}}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j = \sum_{k=1}^{q} \operatorname{Re} d_k(0) \left(\sum_{i,j=1}^{n} \frac{\partial^2 \rho_k}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j \right).$$

Hence if $\xi = \sum_{k=1}^{q}(-\operatorname{Re} d_k(0)) d\rho_k(0) \in N_0(M)$ it follows that L_p^ξ is strictly positive definite and by Proposition 2, p is an extreme point of M, which contradicts the fact that M has no extreme points.

REMARK. Professor Hugo Rossi pointed to me that Theorem 3 can be proved by using the approximation theorem of Baouendi and Trèves **[1]** and the techniques from his paper **[6]**.

ACKNOWLEDGEMENT. I wish to thank the referee for his suggestions which simplified the proof of Theorem 3.

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