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DECOMPOSITION OF POSITIVE PROJECTIONS ONTO JORDAN ALGEBRAS

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ABSTRACT. If a positive projection P from a C^*-algebra onto a Jordan algebra can be decomposed as a sum of maps satisfying certain inequalities of the Schwarz type, then P is actually a sum of completely positive and completely copositive maps.

Suppose that P is a positive unital projection on a C^*-algebra A such that $P(A)_{sa}$ is a JC-algebra. P is said to be *decomposable* if it is the sum of completely positive and completely copositive maps on A. It was shown in [**6**] that P is decomposable if and only if $P(A)_{sa}$ is reversible in the sense that it is closed under symmetric products of the type $x_1x_2 \cdots x_n + x_n \cdots x_2x_1$. Using this result, it was shown in [**5**] that P is decomposable whenever it can be expressed as a sum of 2-positive and 2-copositive maps. We now give an improvement to the latter result, using a technique of [**2**].

If A is a unital C^*-algebra and H is a Hilbert space, then a linear map ϕ: $A \to B(H)$ is called a Schwarz map [**4**] if it satisfies the inequality $\phi(a)^*\phi(a) \leqslant \phi(a^*a)$ for all $a \in A$. Similarly ϕ is an anti-Schwarz map if it satisfies $\phi(a)\phi(a)^* \leqslant \phi(a^*a)$ for all $a \in A$. Every unital 2-positive map ϕ is a Schwarz map, but not conversely [**1**]. Similarly every unital 2-copositive map is an anti-Schwarz map. It follows from [**1**, Lemma 2.1] that a unital linear map ϕ is a Schwarz map if and only if

$$\phi \otimes I_2\left(1 \otimes e_{11} + a \otimes e_{12} + a^* \otimes e_{21} + a^*a \otimes e_{22}\right) \geqslant 0,$$

for all $a \in A$. There is obviously a similar condition for anti-Schwarz maps.

In order to study the decomposition of P we need to consider nonunital maps ϕ. The ideas of [**2**, Lemma 3] are useful here. Let $h = \phi(1)^{1/2}$, with support projection p. Then h^{-1} exists as a positive (unbounded) selfadjoint operator affiliated with $pB(H)p$, and we can define a positive unital linear map $\tilde{\phi}$ on A by $\tilde{\phi}(a) = h^{-1}\phi(a)h^{-1}$. We shall say that ϕ is *sesqui-positive* if $\tilde{\phi}$ is a Schwarz map and that ϕ is *sesqui-copositive* if $\tilde{\phi}$ is an anti-Schwarz map. We then have the following characterization of sesqui-positive maps, which has an obvious analogue for sesqui-copositive maps.

LEMMA 1. *The following statements are equivalent, for a positive linear map* ϕ: $A \to B(H)$.

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(1) ϕ *is sesqui-positive.*

(2) $\phi(a^*a) \geqslant \phi(a)^*\phi(1)^{-1}\phi(a)$, *for all* $a \in A$.

(3) $\phi \otimes I_2(1 \otimes e_{11} + a \otimes e_{12} + a^* \otimes e_{21} + a^*a \otimes e_{22}) \geqslant 0$, *for all* $a \in A$.

PROOF. The right-hand side of the inequality in (2) is to be interpreted as the strong limit of the sequence $\phi(a)^*(\phi(1) + 1/n)^{-1}\phi(a)$ as $n \to \infty$, as in [**3**]. The equivalence of the three conditions follows from [**2**, Lemma 3, **1**, Lemma 2.1]. □

We need two more preliminary lemmas before proving the main result.

LEMMA 2. *If* ϕ: $A \to B(H)$ *is a sesqui-positive (sesqui-copositive) linear map, then the same is true for the second dual map* ϕ^{**}.

PROOF. This follows easily from condition (3) of Lemma 1, by approximating elements of A^{**} in the σ-strong* topology. □

LEMMA 3. *Let M be a von Neumann algebra, B a JW-subalgebra of M_{sa} and ϕ a positive linear map on M such that $\phi(x) \leqslant x$ whenever $0 \leqslant x \in B$. Then $\phi(b) = \phi(1)b = b\phi(1)$ for all $b \in B$.*

PROOF. Given a projection $p \in B$, we have $0 \leqslant \phi(p) \leqslant p$, so that $(1 - p)\phi(p) = 0$. Replacing p by $1 - p$ gives $p\phi(1 - p) = 0$, and subtraction of these two equations results in $\phi(p) = p\phi(1)$. The result follows, since B is the closed linear span of its projections. □

THEOREM 4. *Let A be a unital C^*-algebra and P a positive unital projection on A such that $B = P(A)_{sa}$ is a JC-subalgebra of A. If $P = \phi + \psi$, where ϕ is sesqui-positive and ψ is sesqui-copositive, with $\phi(1)$, $\psi(1)$ invertible, then P is decomposable.*

PROOF. It is enough to show that the second dual map P^{**} is decomposable. We may identify B^{**} with the ultraweak closure of B in the von Neumann algebra A^{**}, so that $B^{**} = P^{**}(A^{**}_{sa})$ is a JW-subalgebra of A [**7**, Theorem 1]. Therefore, by Lemma 2, we may suppose that A is a von Neumann algebra, B is a JW-subalgebra and P is a normal projection satisfying $P(A)_{sa} = B$. To show that P is decomposable, it suffices to show that B is reversible [**6**, Corollary 7.4].

Let $h = \phi(1)^{1/2}$ and $k = \psi(1)^{1/2}$. Define $\tilde{\phi}$ and $\tilde{\psi}$ to be the positive unital linear maps associated with ϕ and ψ respectively. Then $\tilde{\phi}$ is a Schwarz map and $\tilde{\psi}$ is an anti-Schwarz map, and

$$P(a) = h\tilde{\phi}(a)h + k\tilde{\psi}(a)k.$$

Note that $h^2 + k^2 = 1$, from which it follows that h and k commute.

Now if $b \in B$ then $P(b) = b$, so by Lemma 3, $\phi(b) = h^2b = bh^2$ and $\psi(b) = k^2b = bk^2$. In particular $hb = bh$ and $kb = bk$, whenever $b \in B$, so that $\tilde{\phi}(b) = \tilde{\psi}(b) = b$. Since B is a Jordan algebra, we also have $\tilde{\phi}(b^2) = \tilde{\psi}(b^2) = b^2$.

We can now show that B is reversible. We may suppose that A is the von Neumann algebra generated by B. It follows from [**4**, Theorem] and its anti-Schwarz analogue that $\tilde{\phi}$ is the identity map and $\tilde{\psi}$ is an anti-automorphism of order 2 of A

which fixes the elements of B. Also, h, k belong to the centre of A and

$$P(a) = h^2\tilde{\phi}(a) + k^2\tilde{\psi}(a).$$

It is clear from this that B is reversible, which proves the result. □

REMARKS. (1) The argument above proves more than stated. In fact if A is the von Neumann algebra generated by a JW-algebra B and P is a positive unital projection satisfying $P(A_{sa}) = B$ and $P = \phi + \psi$, with ϕ, ψ as before, then

$$P(a) = z_1 a + z_2 \alpha(a),$$

where $z_1, z_2 \geqslant 0$ are in the centre of A, $z_1 + z_2 = 1$ and α is a * antiautomorphism of order 2 of A.

(2) If B is a spin factor of dimension other than 3, 4 or 6, then the corresponding projection P fails to have a decomposition as in [**6**, Theorem 4].

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NOTE ADDED IN PROOF. E. Størmer has kindly drawn the author's attention to a paper of T. B. Andersen, [*On multipliers and order-bounded operators in C^*-algebras*, Proc. Amer. Math. Soc. **25** (1970), 896–899] which proves a C^*-algebraic version of Lemma 3.

REFERENCES

1. M. D. Choi, *Some assorted inequalities for positive linear maps on C^*-algebras*, J. Operator Theory **4** (1980), 271–285.

2. L. T. Gardner, *Linear maps of C^*-algebras preserving the absolute value*, Proc. Amer. Math. Soc. **76** (1979), 271–278.

3. E. H. Lieb and M. Ruskai, *Some operator inequalities of the Schwarz type*, Adv. in Math. **12** (1974), 269–273.

4. T. W. Palmer, *Characterizations of $*$-homomorphisms and expectations*, Proc. Amer. Math. Soc. **46** (1974), 265–272.

5. A. G. Robertson, *Positive projections on C^*-algebras and an extremal positive map*, J. London Math. Soc. (2) **32** (1985), 133–140.

6. E. Størmer, *Decomposition of positive projections on C^*-algebras*, Math. Ann. **274** (1980), 21–41.

7. E. Effros and E. Størmer, *Jordan algebras of self-adjoint operators*, Trans. Amer. Math. Soc. **127** (1967), 313–316.

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