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# AN OPERATOR APPROACH TO THE PRINCIPLE OF INCLUSION AND EXCLUSION

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ABSTRACT. Using an operator approach we derive Sylvester-Whitworth formulae for sets $A$'s. By the same token we treat the problem where both sets of $A$'s and $B$'s are involved. Our result extends the Sylvester-Whitworth inclusion and exclusion formula to the resolution of the number of elements in exactly $m_1$ sets of $A$'s and $m_2$ sets of $B$'s respectively. The formula are applied to the complete graph and complete bipartite graph. The enumeration of spanning subgraphs with any preassigned number of disconnected cycles is solved, together with the case where any preassigned number of vertices have degree one.

**1. Introduction.** Recently, Chow and the present author introduced an operator approach to some enumeration problems in a series of articles [**1**, **2**, **3**, **4**]. The general idea is as follows. First, one constructs a vector space of formal sums on which some operators are defined. Then a real-valued linear function on the formal vector space is introduced. This linear function plays an important role in the whole approach. Whether the operator method is applicable to a particular problem depends on the successful search for such a function in each case. The problem we studied by this approach are enumerations of forests and matchings of a graph and also that of connected spanning subgraphs with preassigned cyclomatic number (of a planar graph). In this article, we use the operator approach to treat the principle of inclusion and exclusion which has been investigated in the past and most recently in [**5**, **6**, **7**]. The vector space of formal sums under consideration is defined as in references [**1**–**4**], but a different real-valued linear function is now introduced as we are treating a different problem here. First, we derive Sylvester-Whitworth formulae for sets of $A$'s. Next, using the same formalism, we treat the problem where both sets of $A$'s and $B$'s are involved. Our result extends the Sylvester-Whitworth inclusion and exclusion formula to the resolution of the number of elements in exactly $m_1$ sets of $A$'s and $m_2$ sets of $B$'s respectively. Finally, we apply the resulting formulae to the complete graph and complete bipartite graph. The enumeration of spanning subgraphs with any preassigned number of disconnected cycles is resolved, together with the case where any preassigned number of vertices have degree one.

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**2. Main theorems.** Let $\mathbf{R} = \{1, \ldots, n, \ldots, r\}$ be a set and $\mathcal{P}(\mathbf{R})$ be the power set of $\mathbf{R}$, i.e., the collection of all subsets in $\mathbf{R}$. Construct the vector space $\mathbf{U}$ of

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the formal sums generated by $\mathcal{P}(\mathbf{R})$. The linear "creation" operator is defined by

$$\lambda_i\colon Q \mapsto \begin{cases} Q \cup \{i\}, & \text{if } i \notin Q, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $Q \subseteq \mathbf{R}$ and the 0 in (1) is the zero formal sum in $\mathbf{U}$.

Let $A_1, \ldots, A_n, \ldots, A_r$ be $r$ sets in the universe $\mathcal{U}$. Denote by $A_iA_j \cdots A_k$ the intersection $A_i \cap A_j \cap \cdots \cap A_k$. Denote by $N(A_iA_j \cdots A_k)$ the number of elements in $A_iA_j \cdots A_k$. Next, we define a linear function $\mu$ on the space $\mathbf{U}$ by

$$\mu\colon Q \mapsto N(A_Q), \qquad A_Q = \prod_{i \in Q} A_i. \tag{2}$$

It follows from (1) and (2) that

$$\lambda_i\lambda_j = \lambda_j\lambda_i, \qquad \lambda_i^2 = 0, \ldots, \; i, j \in \mathbf{R} \tag{3}$$

and

$$N(A_{Q\cup\{i\}}) = N(A_QA_i), \quad \text{if } i \notin Q. \tag{4}$$

LEMMA 1. *Let $S$ be a subset of $R$ and $S \cap \{1, \ldots, n\} = \varnothing$. Then*

$$\mu\left[\prod_{i=1}^{n}(1-\lambda_i)S\right] = N(A_S\overline{A}_1 \cdots \overline{A}_n) \tag{5}$$

*where $\overline{A}_i$ is the complement of $A_i$.*

PROOF. Use mathematical induction on $n$.

$$\begin{aligned} \mu\left[\prod_{i=1}^{n+1}(1-\lambda_i)S\right] &= \mu\left[\prod_{i=1}^{n}(1-\lambda_i)S\right] - \mu\left[\prod_{i=1}^{n}(1-\lambda_i)S \cup \{n+1\}\right] \\ &= N(A_S\overline{A}_1 \cdots \overline{A}_n) - N(A_SA_{n+1}\overline{A}_1 \cdots \overline{A}_n) \\ &= N(A_S\overline{A}_1 \cdots \overline{A}_{n+1}). \end{aligned} \tag{6}$$

The last step follows from the inclusion and exclusion argument.

LEMMA 2. *Let $S$ be a subset of $\mathbf{R}$ and $S \cap \{1, \ldots, n\} = \varnothing$. Then*

$$N_m(A_S) = \mu\left[\prod_{i=1}^{n}(1-\lambda_i)\frac{1}{m!}\lambda^m S\right], \qquad 0 \le m \le n, \tag{7}$$

*where*

$$N_m(A_S) = \sum_{i_1 < \cdots < i_m}^{n} N(A_SA_{i_1} \cdots A_{i_m}\overline{A}_{j_1} \cdots \overline{A}_{j_{n-m}}), \tag{8}$$

$$\{i_1, \ldots, i_m\} \cap \{j_1, \ldots, j_{n-m}\} = \varnothing, \tag{9}$$

$$\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_{n-m}\} = \{1, \ldots, n\} \tag{10}$$

*and*

$$\lambda = \lambda_1 + \cdots + \lambda_n. \tag{11}$$

PROOF. It follows from (3) that

$$\frac{1}{m!}\lambda^m = \sum_{i_1<\cdots<i_m}^{n} \lambda_{i_1}\cdots\lambda_{i_m} \tag{12}$$

and

$$\prod_{i=1}^{n}(1-\lambda_i)\frac{1}{m!}\lambda^m = \sum_{i_1<\cdots<i_m}^{n} \prod_{j\neq i_1,\ldots,i_m}^{n} (1-\lambda_j)\lambda_{i_1}\cdots\lambda_{i_m}. \tag{13}$$

Hence

$$\mu\left[\prod_{i=1}^{n}(1-\lambda_i)\frac{1}{m!}\lambda^m S\right] = \sum_{i_1<\cdots<i_m}^{n} \mu\left[\prod_{j\neq i_1,\ldots,i_m}^{n} (1-\lambda_j) S\cup\{i_1,\ldots,i_m\}\right]. \tag{14}$$

Then Lemma 1 completes the proof.

THEOREM 1. *Let $A_1,\ldots,A_r$ be $r$ sets in a universe $\mathcal{U}$. Let $S$ be a subset of $\mathbf{R}$, $\mathbf{R}=\{1,\ldots,n,\ldots,r\}$, and $S\cap\{1,\ldots,n\}=\varnothing$. Denote by $W_m(A_S)$ the number of the sizes of all $m$-tuple intersections of $A_S$ and $A_i$'s, $i=1,\ldots,n$. Then the number of elements in exactly $m$ of $A_iA_S$, $N_m(A_S)$, is given by*

$$N_m(A_S) = \sum_{k=0}^{n-m}(-1)^k\binom{m+k}{m}W_{m+k}(A_S), \qquad 0\le m\le n, \tag{15}$$

*which is the Sylvester-Whitworth inclusion and exclusion formula.*

PROOF. By (3) it is readily seen that

$$1-\lambda_i = e^{-\lambda_i} \tag{16}$$

and

$$\prod_{i=1}^{n}(1-\lambda_i) = e^{-\lambda}. \tag{17}$$

Then Lemma 2 yields

$$N_m(A_S) = \mu\left[\frac{1}{m!}\lambda^m e^{-\lambda}S\right]. \tag{18}$$

By definition, one reads

$$\mu\left[\frac{1}{m!}\lambda^m S\right] = \sum_{i_1<\cdots<i_m}^{n} N(A_S A_{i_1}\cdots A_{i_m}) = W_m(A_S). \tag{19}$$

The relations of (18) and (19) lead to (15).

REMARKS. (i) It follows from (18) and (19) that

$$\begin{aligned} W_m(A_S) &= \mu\left[\left(\frac{1}{m!}\lambda^m e^{\lambda}\right)e^{-\lambda}S\right] \\ &= \sum_{k=0}^{n-m}\frac{1}{k!m!}\mu[\lambda^{m+k}e^{-\lambda}S] \\ &= \sum_{k=0}^{n-m}\binom{m+k}{m}N_{m+k}(A_S) \end{aligned} \tag{20}$$

which can also be established by combinatorial analysis [**8**].

(ii) The operator approach used here has certain similarity to the symbolic method [**5**]. However, they are not identical since every term in the operator approach can be identified while the symbolic method fails to do so.

Next, we use the same formalism to generalize the Sylvester-Whitworth formula which provides the resolution to the number of elements in $m_1$ sets of $A$'s and in $m_2$ sets of $B$'s (both $A$'s and $B$'s are sets in the universe $\mathcal{U}$). To do this, we proceed as follows. Let $\mathbf{R}_k = \{1, \dots, n_k, \dots, r_k\}$, $k = 1, 2$. Denote by $\mathcal{P}(\mathbf{R}_1 \times \mathbf{R}_2)$ the power set of the Cartesian product $\mathbf{R}_1 \times \mathbf{R}_2$. Construct the vector space $\mathbf{U}^2$ of the formal sums generated by $\mathcal{P}(\mathbf{R}_1 \times \mathbf{R}_2)$ with real coefficients. The linear operators $\lambda_i^{(k)}$, $k = 1, 2$, are defined by

$$\lambda_i^{(k)}: (Q_1, Q_2) \mapsto \begin{cases} (Q_1 \cup \{i\}, Q_2), & \text{if } k = 1 \text{ and } i \notin Q_1, \\ (Q_1, Q_2 \cup \{i\}), & \text{if } k = 2 \text{ and } i \notin Q_2, \\ 0, \text{ otherwise}, \end{cases} \tag{21}$$

where $Q_k \subseteq \mathbf{R}_k$, $k = 1, 2$, and 0 in (21) is the zero formal sum in $\mathbf{U}^2$.

It follows from (21)

$$\lambda_i^{(k)} \lambda_j^{(k)} = \lambda_j^{(k)} \lambda_i^{(k)}, \qquad \lambda_i^{(k)} \lambda_i^{(k)} = 0, \qquad k = 1, 2, \tag{22}$$

and

$$\lambda_i^{(1)} \lambda_j^{(2)} = \lambda_j^{(2)} \lambda_i^{(1)}. \tag{23}$$

Let $A_1, \dots, A_{n_1}, \dots, A_{r_1}$, $B_1, \dots, B_{n_2}, \dots, B_{r_2}$ be $(r_1 + r_2)$ sets in the universe $\mathcal{U}$. Define a linear function $\mu$ on the space $\mathbf{U}^2$ by

$$\mu: (Q_1, Q_2) \mapsto N(A_{Q_1} B_{Q_2}) \tag{24}$$

where

$$A_{Q_1} = \prod_{i \in Q_1} A_i \quad \text{and} \quad B_{Q_2} = \prod_{i \in Q_2} B_i. \tag{25}$$

It is readily seen that

$$N(A_{Q_1 \cup \{i\}} B_{Q_2 \cup \{j\}}) = N(A_{Q_1} B_{Q_2} A_i B_j), \tag{26}$$

provided $i \in \mathbf{R}_1$, $i \notin Q_1$, $j \in \mathbf{R}_2, j \notin Q_2$.

The following result can be established easily by the same argument used in the proof of Lemma 1.

LEMMA 3. *Let $S_k$ be a subset of $\mathbf{R}_k$ and $S_k \cap \{1, \dots, n_k\} = \varnothing$, $k = 1, 2$. Then*

$$\mu \left[ \prod_{k=1}^{2} \prod_{i=1}^{n_k} (1 - \lambda_i^{(k)})(S_1, S_2) \right] = N \left( A_{S_1} B_{S_2} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} \overline{A}_i \overline{B}_j \right). \tag{27}$$

LEMMA 4. *Let $S_k$ be a subset of $\mathbf{R}_k$ and $S_k \cap \{1, \dots, n_k\} = \varnothing$, $k = 1, 2$. Then*

$$N_{m_1, m_2}(A_{S_1} B_{S_2}) = \mu \left[ \prod_{k=1}^{2} \prod_{i=1}^{n_k} \frac{1}{m_k!} (1 - \lambda_i^{(k)})(\lambda^{(k)})^{m_k} (S_1, S_2) \right] \tag{28}$$

*where*

$$(29)\quad N_{m_1,m_2}(A_{S_1}B_{S_2}) = \sum_{i_1<\cdots<i_{m_1}}^{n_1}\sum_{j_1<\cdots<j_{m_2}}^{n_2} N(A_{S_1}B_{S_2}A_{i_1}\cdots A_{i_{m_1}}B_{j_1} \cdots B_{j_{m_2}}\overline{A}_{k_1}\cdots\overline{A}_{k_{n_1-m_1}}\overline{B}_{l_1}\cdots\overline{B}_{l_{n_2-m_2}}),$$

$$(30)\quad \{i_1,\ldots,i_{m_1}\}\cap\{k_1,\ldots,k_{n_1-m_1}\}=\varnothing,$$

$$(31)\quad \{i_1,\ldots,i_{m_1}\}\cup\{k_1,\ldots,k_{n_1-m_1}\}=\{1,\ldots,n_1\},\quad \textit{etc.}$$

*and*

$$(32)\quad \lambda^{(k)}=\lambda_1^{(k)}+\cdots+\lambda_{n_k}^{(k)},\quad k=1,2.$$

PROOF. It follows from (22) and (23)

$$(33)\quad \frac{1}{m_k!}(\lambda^{(k)})^{m_k} = \sum_{i_1<\cdots<i_{m_1}}^{n_k}\lambda_{i_1}^{(k)}\cdots\lambda_{i_{m_k}}^{(k)},\qquad k=1,2.$$

Hence

$$(34)\quad \mu\left[\prod_{k=1}^{2}\prod_{l=1}^{n_k}\frac{1}{m_k!}(1-\lambda_l^{(k)})(\lambda^{(k)})^{m_k}(S_1,S_2)\right]$$
$$= \sum_{i_1<\cdots<i_{m_1}}^{n_1}\sum_{j_1<\cdots<j_{m_2}}^{n_2}\mu\left[\prod_{i\neq i_1,\ldots,i_{m_1}}^{n_1}\prod_{j\neq j_1,\ldots,j_{m_2}}^{n_2}(1-\lambda_i^{(1)}) \cdot(1-\lambda_j^{(2)})\lambda_{i_1}^{(1)}\cdots\lambda_{i_{m_1}}^{(1)}\lambda_{j_1}^{(2)}\cdots\lambda_{j_{m_2}}^{(2)}(S_1,S_2)\right]$$
$$= \sum_{i_1<\cdots<i_{m_1}}^{n_1}\sum_{j_1<\cdots<j_{m_2}}^{n_2}\mu\left[\prod_{i\neq i_1,\ldots,i_{m_1}}^{n_1}\prod_{j\neq j_1,\ldots,j_{m_2}}^{n_2}(1-\lambda_i^{(1)}) \cdot(1-\lambda_j^{(2)})(S_1\cup\{i_1,\ldots,i_{m_1}\},S_2\cup\{j_1,\ldots,j_{m_2}\})\right].$$

Lemma 3 completes the proof.

THEOREM 2. *Let $A_1,\ldots,A_{n_1},\ldots,A_{r_1},B_1,\ldots,B_{n_2},\ldots,B_{r_2}$ be $(r_1+r_2)$ sets in a universe $\mathcal{U}$. Let $S_k$ be a subset of $\mathbf{R}_k$ with $\mathbf{R}_k=\{1,\ldots,n_k,\ldots,r_k\}$ and $S_k\cap\{1,\ldots,n_k\}=\varnothing$, $k=1,2$. Then*

$$(35)\quad N_{m_1,m_2}(A_{S_1}B_{S_2}) = \sum_{k_1=0}^{n_1-m_1}\sum_{k_2=0}^{n_2-m_2}(-1)^{k_1+k_2}\binom{m_1+k_1}{m_1}\binom{m_2+k_2}{m_2} \cdot W_{m_1+k_1,m_2+k_2}(A_{S_1}B_{S_2}),$$

*where*

$$(36)\quad W_{m_1,m_2}(A_{S_1}B_{S_2}) = \sum_{i_1<\cdots<i_{m_1}}^{n_1} \sum_{j_1<\cdots<j_{m_2}}^{n_2} N(A_{S_1}B_{S_2}A_{i_1}\cdots A_{i_{m_1}}B_{j_1}\cdots B_{j_{m_2}})$$

*and* $N_{m_1,m_2}(A_{S_1}B_{S_2})$ *is the number of elements in exactly* $m_1$ *of* $A_iA_{S_1}$, $i = 1,\ldots,n_1$, *and* $m_2$ *of* $B_jB_{S_2}$, $j = 1,\ldots,n_2$, *respectively.*

PROOF. By (22) and (23) one finds

$$(37)\qquad \prod_{i=1}^{n_k}(1-\lambda_i^{(k)}) = e^{-\lambda^{(k)}}, \qquad k = 1,2,$$

where $\lambda^{(k)}$ is defined by (32). It follows from Lemma 4

$$(38)\qquad N_{m_1,m_2}(A_{S_1}B_{S_2}) = \mu\left[\prod_{k=1}^{2}\frac{1}{m_k!}(\lambda^{(k)})^{m_k}e^{-\lambda^{(k)}}(S_1,S_2)\right].$$

By definition, we have

$$(39)\qquad \begin{aligned} &\mu\left[\prod_{k=1}^{2}\frac{1}{m_k!}(\lambda^{(k)})^{m_k}(S_1,S_2)\right] \\ &\quad = \sum_{i_1<\cdots<i_{m_1}}^{n_1}\sum_{j_1<\cdots<j_{m_2}}^{n_2} N(A_{S_1}B_{S_2}A_{i_1}\cdots A_{i_{m_1}}B_{j_1}\cdots B_{j_{m_2}}) \\ &\quad = W_{m_1,m_2}(A_{S_1}B_{S_2}). \end{aligned}$$

The theorem is established by (38) and (39).

**3. Spanning subgraphs with disconnected cycles and vertices of degree one.** For a complete graph $K_n$, let $\rho_d(n)$ be the number of $d$-component spanning unicyclic subgraphs with cycle length $l_1,\ldots,l_d$ respectively. For $d=1$, it is known [**9**]

$$(40)\qquad \rho_1(n) = \tfrac{1}{2}(n)_l n^{n-l-1}.$$

In fact, (40) can be established easily by considering all spanning subgraphs with a cycle of cycle length $l$ attaching to a $l$-component forests which can be enumerated by the Kirchoff matrix [**1**, **4**, **10**], i.e.,

$$(41)\qquad \rho_1(n) = \binom{n}{l}\tfrac{1}{2}(l-1)![ln^{n-l-1}].$$

Using the same arguments, it is readily seen that

$$(42)\qquad \rho_d(n) = \frac{l}{2^d}\left(\prod_{i=1}^{d} l_i\right)^{-1}(n)_l n^{n-l-1},$$

where

$$(43)\qquad l = l_1 + \cdots + l_d.$$

For a complete bipartite graph $K_{n_1,n_2}$, let $\rho_d(n_1,n_2)$ be the number of $d$-component spanning unicyclic subgraphs with cycle length $2l_1,\ldots,2l_d$ respectively. A straightforward calculation yields

$$\rho_d(n_1,n_2)=\frac{l}{2^d}\left(\prod_{i=1}^{d}l_i\right)^{-1}(n_1+n_2-l)(n_1)_l(n_2)_l n_1^{n_2-l-1}n_2^{n_1-l-1} \tag{44}$$

where

$$l=l_1+\cdots+l_d. \tag{45}$$

We now consider the application of Theorems 1 and 2. First, we take up the complete graph $K_n$. Let $A_i$ be the set of spanning subgraphs of $K_n$ with vertex $i$ having degree one. Let $A_S$ be the set of a $d$-component, $d=|S|$, spanning unicyclic subgraphs of $K_n$. It is readily seen that

$$\begin{aligned} N(A_SA_{i_1}\cdots A_{i_k}) &= \prod_{j=1}^{k}(\text{number of edges containing vertex } i_j)\rho_d(n-k) \\ &= (n-k)^k\rho_d(n-k). \end{aligned} \tag{46}$$

It follows from (20)

$$W_k(A_S)=\binom{n}{k}(n-k)^k\rho_d(n-k). \tag{47}$$

Denote by $N_m^{(n)}(l_1,\ldots,l_d)$ the number of $d$-component spanning unicyclic subgraphs of $K_n$ with $m$ vertices of degree one and $d$ disconnected cycles each of which have cycle length $l_1,\ldots,l_d$ respectively. Theorem 1 and (47) lead to

$$N_m^{(n)}(l_1,\ldots,l_d)=\frac{n!}{m!}\sum_{k=m}^{n-l}(-1)^{k+m}\frac{(n-k)^k}{(n-k)!(k-m)!}\rho_d(n-k). \tag{48}$$

It follows from (42)

$$N_m^{(n)}(l_1,\ldots,l_d)=\frac{l}{2^d}\left(\prod_{i=1}^{d}l_i\right)^{-1}\frac{n!}{m!}\sum_{k=0}^{n-l-1}l^k\binom{n-l-1}{k}\left\{\begin{matrix}n-l-k-1\\ n-m-l\end{matrix}\right\} \tag{49}$$

where $\left\{\begin{smallmatrix}a\\ b\end{smallmatrix}\right\}$ is the Stirling number of the second kind.

Second, we consider the complete bipartite graph $K_{n_1,n_2}$. Let $A_i$ be the set of spanning subgraphs of $K_{n_1,n_2}$ with vertex $i$, $i=1,\ldots,n_1$, having degree one and $B_j$ be the set of spanning subgraphs of $K_{n_1,n_2}$ with vertex $j$, $j=1,\ldots,n_2$, having degree one. Let $A_{S_1}B_{S_2}$ be the set of a $d$-component, $d=|S_1|=|S_2|$, spanning unicyclic subgraphs of $K_{n_1,n_2}$. A calculation similar to that of the complete graph $K_n$ yields the following result

$$W_{k_1,k_2}(A_{S_1}B_{S_2})=\binom{n_1}{k_1}\binom{n_2}{k_2}(n_1-k_1)^{k_2}(n_2-k_2)^{k_1}\rho_d(n_1,n_2). \tag{50}$$

Denote by $N_{m_1,m_2}^{(n_1,n_2)}(2l_1,\ldots,2l_d)$ the number of $d$-component spanning unicyclic subgraphs of $K_{n_1,n_2}$ with $m_1$, among $n_1$, vertices having degree one in one part and $m_2$, among $n_2$, vertices having degree one in another part, together with $d$ disconnected cycles each of which have cycle length $2l_1,\ldots,2l_d$ respectively. Theorem

2 and (50) lead to

$$(51)\quad N_{m_1,m_2}^{(n_1,n_2)}(2l_1,\ldots,2l_d) = \frac{n_1!n_2!}{m_1!m_2!}\sum_{k_1=m_1}^{n_1-l}\sum_{k_2=m_2}^{n_2-l}\frac{(n_1-k_1)^{k_2}(n_2-k_2)^{k_1}}{(n_1-k_1)!(k_1-m_1)!(n_2-k_2)!(k_2-m_2)!} \cdot(-1)^{k_1+k_2+m_1+m_2}\rho_d(n_1-k_1,n_2-k_2).$$

Using (44), we have

$$(53)\quad N_{m_1,m_2}^{(n_1,n_2)}(2l_1,\ldots,2l_d) = \frac{l}{2^d}\left(\prod_{j=1}^{d} l_i\right)^{-1}\frac{n_1!n_2!}{m_1!m_2!}[D_{m_1}(n_1;n_2;l)D_{m_2}(n_2;n_1-1;l) + D_{m_1}(n_1;n_2-1;l)D_{m_2}(n_2;n_1;l) - D_{m_1}(n_1;n_2-1;l)D_{m_2}(n_2;n_1-1;l)]$$

and

$$l = l_1+\cdots+l_d$$

where

$$(54)\qquad D_m(a;b;c)\equiv\sum_{k=0}^{b-c}c^k\binom{b-c}{k}\left\{\begin{matrix} b-c-k \\ a-c-m\end{matrix}\right\}.$$

REMARK. The relations of (48) and (51) can also be used in other enumeration problems. For example, let $\rho_0(n)$ be the number of spanning trees of the complete graph $K_n$. Denote by $N_m^{(n)}$ the number of spanning trees of $K_n$ with $m$ vertices having degree one. Then

$$(55)\qquad N_m^{(n)} = \frac{n!}{m!}\left\{\begin{matrix} n-2 \\ n-m\end{matrix}\right\}$$

is the result of (48). Historically, Rényi obtained (55) from a recurrence relation **[11]**

$$(56)\qquad \frac{m}{n}N_m^{(n)} = (n-m)N_{m-1}^{(n-1)} + mN_m^{(n-1)}.$$

Next, let $\rho_0(n_1,n_2)$ be the number of spanning trees of the complete bipartite graph $K_{n_1,n_2}$. Denote by $N_{m_1,m_2}^{(n_1,n_2)}$ the number of spanning trees of $K_{n_1,n_2}$ with $m_1$, among $n_1$, vertices having degree one in one part and $m_2$, among $n_2$, vertices having degree one in another part respectively. It follows from (51)

$$(57)\qquad N_{m_1,m_2}^{(n_1,n_2)} = \frac{n_1!n_2!}{m_1!m_2!}\left\{\begin{matrix} n_1-1 \\ n_2-m_2\end{matrix}\right\}\left\{\begin{matrix} n_2-1 \\ n_1-m_1\end{matrix}\right\}.$$

We announce here (to be published elsewhere) the following result concerning the case with sets $A_1^{(i)}, \dots, A_{n_i}^{(i)}, \dots, A_{r_i}^{(i)}$, $i = 1, \dots, p$, in the universe $\mathcal{U}$:

$$N_{m_1,\dots,m_p}\left(\prod_{j=1}^{p} A_{S_j}^{(j)}\right) = \prod_{i=1}^{p}\left[\sum_{k_i=0}^{n_i-m_i}(-1)^{k_i}\binom{m_i+k_i}{m_i}\right] \cdot W_{m_1+k_1,\dots,m_p+k_p}\left(\prod_{j=1}^{p} A_{S_j}^{(j)}\right). \tag{58}$$

We note that (58) is equivalent to Theorem 4 in reference [**12**].

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