POLE- AND ZERO-FREE REGIONS FOR ANALYTIC CONTINUED FRACTIONS

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ABSTRACT. By using well-known methods of analytic continued fraction theory, various types of zero-free regions are obtained for sequences of polynomials having complex coefficients and being defined by three-term recurrence relations. These results are related to recent investigations by P. Henrici, E. B. Saff and R. S. Varga. As an application, zero-free sectors and stripes in $\mathbb{C}$ are obtained for the Bessel function J_ν , where ν is complex. Analogous results are obtained for the Lommel polynomials associated with J_ν .

1. Introduction. Consider the continued fraction

$$(1) \quad w_n = \frac{A_n}{B_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}, \quad n \in \mathbb{N},$$

where $a_n, b_n \in \mathbb{C}$ and $a_n \neq 0$ for $n \in \mathbb{N}$. In the applications below, a_n and b_n are special analytic functions of z . Then (see [2, p. 20])

$$(2) \quad \begin{aligned} B_n &= b_n B_{n-1} + a_n B_{n-2}, & B_0 &= 1, & B_{-1} &= 0, & n &\in \mathbb{N}, \\ A_n &= b_n A_{n-1} + a_n A_{n-2}, & A_0 &= 0, & A_{-1} &= 1, & n &\in \mathbb{N}, \end{aligned}$$

hold.

Because $A_n B_{n-1} - B_n A_{n-1} = (-1)^{n-1} a_1 a_2 \cdots a_n$, $n \in \mathbb{N}$, A_n and B_n do not vanish simultaneously, and, hence, $w_n = \infty$ holds iff $B_n = 0$. If a sequence B_n , $n \in \mathbb{N}$, is given which satisfies (2), then, after choosing $a_1 \in \mathbb{C}$, $a_1 \neq 0$, arbitrarily, B_n can be considered as the n th denominator of a sequence of continued fractions $w_n = A_n/B_n$ of type (1).

We want to find conditions on a_n, b_n which imply $w_n \neq \infty$ and, hence, $B_n \neq 0$ for $n \in \mathbb{N}$. Since, for $n \geq 2$, $w_n = a_1/(b_1 + w_{n-1}^*)$ holds, where

$$w_{n-1}^* = \frac{A_{n-1}^*}{B_{n-1}^*} = \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n},$$

$A_n = 0$ holds iff $B_{n-1}^* = 0$ holds.

Hence, results concerning the nonvanishing of B_n , $n \in \mathbb{N}$ also yield results concerning the nonvanishing of A_n , $n \in \mathbb{N}$. In [4] the special case $a_n = -\alpha_n z$, $b_n = \beta_n + z$, $\alpha_n > 0$, $\beta_n > 0$, $n \in \mathbb{N}$, is considered, and zero-free parabolic regions are obtained for the polynomials $B_n(z)$. In [1] also zero-free parabolic regions are obtained for interpolation polynomials $B_n(z)$, where $a_n = -\alpha_n z_n$, $b_n = \beta_n + z_{n+1}$,

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$\alpha_n > 0$, $\beta_n > 0$, and $z_n = z - X_{n-1}$. Here X_n , $n \in \mathbb{N}$, are the real interpolation points. The results of [1 and 4] are generalized in [3] to sequences B_n , $n \in \mathbb{N}$, with complex α_n and β_n .

We first derive general conditions on a_n , b_n which imply $B_n \neq 0$ for $n \in \mathbb{N}$ (Lemma 1). If a_n and b_n are special polynomials in z of degree ≤ 2 , then Lemma 1 yields parabolic regions, sectors and stripes in $\mathbb{C}$ as zero-free regions for $B_n(z)$. In particular, zero-free regions of these types are obtained for the Bessel function J_ν and the corresponding Lommel polynomials, where ν is complex.

2. Main results. As in [1 and 3] the principal idea is to write (1) in the form $w_N = s_1 \circ \dots \circ s_N(0)$, $N \in \mathbb{N}$, where $s_n(u) := a_n/(b_n + u)$, $u \in \mathbb{C}$, $n \in \mathbb{N}$, and to determine a sequence of closed half-planes $H_n \subset \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $0 \in H_n$, $n \in \mathbb{N}$, such that, for $n \in \mathbb{N}$, $D_n := s_n(H_n)$ is a finite closed disk satisfying $D_n \subset H_{n-1}$ for $n \geq 2$. Then $w_N \in D_1$ and, hence, $w_N \neq \infty$ holds for $N \in \mathbb{N}$.

For $n \in \mathbb{N}$ define $H_n := \{z \in \mathbb{C}: \operatorname{Re} e^{i\varphi_n}(z + d_n) \geq 0\} \cup \{\infty\}$, where $\varphi_n \in \mathbb{R}$ and $d_n \in \mathbb{C}$. Then $0 \in H_n$ if $\operatorname{Re} e^{i\varphi_n} d_n \geq 0$, and $D_n = s_n(H_n)$ is a finite disk iff $-b_n \notin H_n$; i.e., $\operatorname{Re} e^{i\varphi_n}(b_n - d_n) > 0$.

Denote by b_n^* the point symmetric to $-b_n$ with respect to ∂H_n ; that is, $b_n^* = -d_n + e^{-2i\varphi_n}(\bar{b}_n - \bar{d}_n)$. Then because $s_n(-b_n) = \infty$, the center ζ_n of D_n is given by

$$\zeta_n = s_n(b_n^*) = \frac{a_n}{b_n + b_n^*} = \frac{a_n e^{i\varphi_n}}{2 \operatorname{Re} e^{i\varphi_n}(b_n - d_n)}.$$

Since $s_n(\infty) = 0 \in \partial D_n$, the radius r_n of D_n is given by

$$r_n = |\zeta_n| = |a_n|/2 \operatorname{Re} e^{i\varphi_n}(b_n - d_n).$$

Now $D_n \subset H_{n-1}$ holds iff $\operatorname{Re} e^{i\varphi_{n-1}}(\zeta_n + d_{n-1}) - r_n \geq 0$ holds ($n \geq 2$). This is equivalent to

$$|a_n| \leq \operatorname{Re} e^{i(\varphi_{n-1} + \varphi_n)} a_n + 2(\operatorname{Re} e^{i\varphi_{n-1}} d_{n-1})(\operatorname{Re} e^{i\varphi_n}(b_n - d_n)), \quad n \geq 2.$$

We have thus proved (see also [2, Chapter 4])

LEMMA 1. Assume that a_n and b_n in (1) or (2) satisfy (for suitable $\varphi_n \in \mathbb{R}$, $d_n \in \mathbb{C}$)

$$(3) \quad \operatorname{Re} e^{i\varphi_n}(b_n - d_n) > 0, \quad \operatorname{Re} e^{i\varphi_n} d_n \geq 0, \quad n \in \mathbb{N},$$

and

$$(4) \quad |a_n| \leq \operatorname{Re} e^{i(\varphi_{n-1} + \varphi_n)} a_n + 2(\operatorname{Re} e^{i\varphi_{n-1}} d_{n-1})(\operatorname{Re} e^{i\varphi_n}(b_n - d_n)), \quad n \geq 2.$$

Then $B_n \neq 0$ for $n \in \mathbb{N}$. If (3) holds for $1 \leq n \leq N$ and (4) holds for $2 \leq n \leq N$, then $B_n \neq 0$ for $1 \leq n \leq N$.

REMARK. In (3) the condition $\operatorname{Re} e^{i\varphi_1}(b_1 - d_1) > 0$ can be replaced by the weaker condition $-b_1 \notin D_2$ —i.e.

$$|2b_1 \operatorname{Re} e^{i\varphi_2}(b_2 - d_2) + a_2 e^{i\varphi_2}| > |a_2|.$$

We now consider special cases of (1) and Lemma 1.

THEOREM 1. *Let*

$$w_n(z) = \frac{A_n(z)}{B_n(z)} = \frac{1}{1} + \frac{c_1 c_2 z}{1} + \frac{c_2 c_3 z}{1} + \dots + \frac{c_{n-1} c_n z}{1},$$

$$B_n(z) = B_{n-1}(z) + c_{n-1} c_n z B_{n-2}(z), \quad n \geq 1, \quad B_0 = 1, \quad B_{-1} = 0,$$

where $c_n \in \mathbb{C}$, $c_n = |c_n|e^{i\psi_n} \neq 0$, $\alpha \leq \psi_n < \beta$, $\beta - \alpha < \pi$, $n \in \mathbb{N}$.

(a) Then $B_n(z) \neq 0$ for $n \geq 1$ and all $z = |z|e^{i\vartheta}$ satisfying $-\pi - 2\alpha < \vartheta \leq \pi - 2\beta$.

(b) If $I := \{\varphi \in \mathbb{R}: \beta - \pi/2 \leq \varphi < \alpha + \pi/2\}$,

$$r(\varphi) := \inf_{n \geq 2} \operatorname{Re}(e^{i\varphi}/c_{n-1}) \operatorname{Re}(e^{i\varphi}/c_n) \quad \text{for } \varphi \in I,$$

and if, for $r \geq 0$, $P(r, \varphi) := \{z \in \mathbb{C}: |z| \leq \operatorname{Re} e^{2i\varphi} z + r/2\}$, then $B_n(z) \neq 0$ for $n \geq 1$ and all $z \in \bigcup_{\varphi \in I} P(r(\varphi), \varphi)$.

PROOF. Put $a_1 = 1$, $a_n = c_{n-1} c_n z$, $n \geq 2$, $b_n = 1$, $n \geq 1$, in (1) and apply Lemma 1. (a) Choose $d_n = 0$ and $e^{i\varphi_n} = e^{-i\vartheta/2} |c_n|/c_n$, $n \geq 1$. Then (4) is trivially satisfied, and (3) reduces to $\operatorname{Re} e^{i\vartheta/2} c_n > 0$, $n \geq 1$, which is satisfied if $-\pi/2 < \psi_n + \vartheta/2 < \pi/2$, $n \geq 1$, or $-\pi - 2\alpha < \vartheta \leq \pi - 2\beta$.

(b) Choose $d_n = 1/2$ and $e^{i\varphi_n} = e^{i\varphi} |c_n|/c_n$, $n \geq 1$. Then (3) and (4) reduce to $\operatorname{Re} e^{i\varphi}/c_n > 0$ and $|z| \leq \operatorname{Re} e^{2i\varphi} z + \frac{1}{2} \operatorname{Re}(e^{i\varphi}/c_{n-1}) \operatorname{Re}(e^{i\varphi}/c_n)$. $\operatorname{Re} e^{i\varphi}/c_n > 0$ is satisfied if $-\pi/2 < \varphi - \psi_n < \pi/2$, $n \geq 1$, or $\beta - \pi/2 \leq \varphi < \alpha + \pi/2$.

THEOREM 2. *Let*

$$w_n(z) = \frac{A_n(z)}{B_n(z)} = \frac{1}{1 + e_1 z} + \frac{c_1 c_2 z^2}{1 + e_2 z} + \frac{c_2 c_3 z^2}{1 + e_3 z} + \dots + \frac{c_{n-1} c_n z^2}{1 + e_n z},$$

$$B_n(z) = (1 + e_n z) B_{n-1}(z) + c_{n-1} c_n z^2 B_{n-2}(z), \quad n \geq 1, \quad B_0 = 1, \quad B_{-1} = 0,$$

where $e_n, c_n \in \mathbb{C}$ and $c_n = |c_n|e^{i\psi_n} \neq 0$, $\alpha \leq \psi_n < \beta$, $\beta - \alpha < \pi$ for $n \in \mathbb{N}$.

(a) If $\operatorname{Re}(e_n/c_n) \geq 0$ (≤ 0), $n \geq 1$, then $B_n(z) \neq 0$ for $n \geq 1$ and all $z = |z|e^{i\vartheta}$ satisfying $-\alpha - \pi/2 < \vartheta \leq -\beta + \pi/2$ ($-\alpha + \pi/2 < \vartheta \leq -\beta + 3\pi/2$).

(b) If $e_n = 0$, $n \geq 1$, and $I, r(\varphi)$ are the same as in Theorem 1, and if, for $r \geq 0$, $S(r, \varphi) := \{z \in \mathbb{C}: |\operatorname{Im} e^{i\varphi} z| \leq \sqrt{r}/2\}$, then $B_n(z) \neq 0$ for $n \geq 1$ and all $z \in \bigcup_{\varphi \in I} S(r(\varphi), \varphi)$.

PROOF. Put $a_1 = 1$, $a_n = c_{n-1} c_n z^2$, $n \geq 2$, $b_n = 1 + e_n z$, $n \geq 1$, in (1) and apply Lemma 1. (a) Choose $d_n = e_n z$ and $e^{i\varphi_n} = e^{-i\vartheta} |c_n|/c_n$, $n \geq 1$. Then (4) is trivially satisfied, and (3) reduces to $\operatorname{Re}(e_n/c_n) \geq 0$ and $\operatorname{Re} z c_n > 0$, $n \geq 1$, which is satisfied if $-\pi/2 < \psi_n + \vartheta < \pi/2$, $n \geq 1$, or $-\alpha - \pi/2 < \vartheta \leq -\beta + \pi/2$. Similarly, $e^{i\varphi_n} = -e^{-i\vartheta} |c_n|/c_n$ yields the conditions $\operatorname{Re}(e_n/c_n) \leq 0$ and $\operatorname{Re}(-z c_n) > 0$ or $-\pi/2 < -\pi + \psi_n + \vartheta < \pi/2$, $n \geq 1$, which is satisfied if $-\alpha + \pi/2 < \vartheta \leq -\beta + 3\pi/2$.

(b) Put $d_n = 1/2$ and $e^{i\varphi_n} = e^{i\varphi} |c_n|/c_n$. Then (4) and (3) reduce to

$$|z^2| \leq \operatorname{Re}(e^{i\varphi} z)^2 + \operatorname{Re}(e^{i\varphi}/c_{n-1}) \operatorname{Re}(e^{i\varphi}/c_n)/2$$

or $2|\operatorname{Im} e^{i\varphi} z| \leq (\operatorname{Re}(e^{i\varphi}/c_{n-1}) \operatorname{Re}(e^{i\varphi}/c_n))^{1/2}$, $n \geq 2$, and $\operatorname{Re} e^{i\varphi}/c_n > 0$, $n \geq 1$, which is satisfied if $-\pi/2 < \varphi - \psi_n < \pi/2$, $n \geq 1$ or $\beta - \pi/2 \leq \varphi < \alpha + \pi/2$.

THEOREM 3. *Let*

$$w_n(z) = \frac{A_n(z)}{B_n(z)} = \frac{1}{g_1 z + h_1} + \frac{1}{g_2 z + h_2} + \cdots + \frac{1}{g_n z + h_n},$$

$$B_n(z) = (g_n z + h_n) B_{n-1}(z) + B_{n-2}(z), \quad n \geq 1, \quad B_0 = 1, \quad B_{-1} = 0,$$

where $g_n, h_n \in \mathbb{C}$, $g_n = |g_n|e^{i\psi_n} \neq 0$, $\alpha \leq \psi_n < \beta$, $\beta - \alpha < \pi$ for $n \in \mathbb{N}$.

(a) If $\operatorname{Re} h_n \geq 0$ (≤ 0), $n \geq 1$, then $B_n(z) \neq 0$ for $n \geq 1$ and all $z = |z|e^{i\vartheta} \neq 0$ satisfying $-\alpha - \pi/2 < \vartheta \leq -\beta + \pi/2$ ($-\alpha + \pi/2 < \vartheta \leq -\beta + 3\pi/2$).

(b) If $h_n = 0$, $n \geq 1$,

$$J := \{\varphi \in \mathbb{R} : -\alpha - \pi/2 < \varphi \leq -\beta + \pi/2\},$$

$$s(\varphi) := \inf_{n \geq 2} \operatorname{Re}(e^{i\varphi} g_{n-1}) \operatorname{Re}(e^{i\varphi} g_n)$$

for $\varphi \in J$, and if

$$D(0, \varphi) := \{z \in \mathbb{C} : \operatorname{Im} e^{-i\varphi} z = 0\}$$

and

$$D(s, \varphi) := \{z \in \mathbb{C} : |z - ie^{i\varphi}/\sqrt{s}| \geq 1/\sqrt{s}\} \cap \{z \in \mathbb{C} : |z + ie^{i\varphi}/\sqrt{s}| \geq 1/\sqrt{s}\}$$

for $s > 0$, then $B_n(z) \neq 0$ for $n \geq 1$ and all $z \in \bigcup_{\varphi \in J} D(s(\varphi), \varphi) \setminus \{0\}$.

PROOF. Put $a_n = 1$, $b_n = g_n z + h_n$, $n \geq 1$, in (1) and apply Lemma 1.

(a) Choose $d_n = 0$, $n \geq 1$. Then (3) and (4) reduce to $\operatorname{Re} e^{i\varphi_n}(g_n z + h_n) > 0$, $n \geq 1$, and $1 \leq \cos(\varphi_{n-1} + \varphi_n)$, $n \geq 2$. This holds if $\varphi_n = 0$, $\operatorname{Re} h_n \geq 0$ and $\operatorname{Re} z g_n > 0$ or $-\pi/2 < \vartheta + \psi_n < \pi/2$, $n \geq 1$, which is satisfied if $-\alpha - \pi/2 < \vartheta \leq -\beta + \pi/2$. Similarly, $\varphi_n = \pi$, $n \geq 1$, yields the conditions $\operatorname{Re} h_n \leq 0$ and $\operatorname{Re}(-z g_n) > 0$ or $-\pi/2 < -\pi + \vartheta + \psi_n < \pi/2$, $n \geq 1$, which is satisfied if $-\alpha + \pi/2 < \vartheta \leq -\beta + 3\pi/2$.

(b) Put $d_n = z g_n/2$ and $e^{i\varphi_n} = e^{i\varphi}|z|/z$, $n \geq 1$. Then (4) and (3) reduce to

$$1 \leq \operatorname{Re}(\bar{z} e^{i\varphi}/|z|)^2 + |z|^2 \operatorname{Re}(e^{i\varphi} g_{n-1}) \operatorname{Re}(e^{i\varphi} g_n)/2, \quad n \geq 2,$$

and

$$\operatorname{Re}(e^{i\varphi} g_n) > 0 \quad \text{or} \quad -\pi/2 < \varphi + \psi_n < \pi/2, \quad n \geq 1,$$

which is satisfied if $-\alpha - \pi/2 < \varphi \leq -\beta + \pi/2$. Finally, one verifies that for $s \geq 0$,

$$\{z \in \mathbb{C} : |z|^2 \leq \operatorname{Re}(z e^{-i\varphi})^2 + |z|^4 s/2\} = D(s, \varphi).$$

REMARK. If the assumptions in Theorems 1–3 hold for $1 \leq n \leq N$ only, then the conclusions are valid for $1 \leq n \leq N$, provided $r(\varphi)$, $s(\varphi)$ are replaced by

$$r_N(\varphi) := \min_{2 \leq n \leq N} \operatorname{Re}\left(\frac{e^{i\varphi}}{c_{n-1}}\right) \operatorname{Re}\left(\frac{e^{i\varphi}}{c_n}\right) \quad \text{for } \varphi \in I,$$

and

$$s_N(\varphi) := \min_{2 \leq n \leq N} \operatorname{Re}(e^{i\varphi} g_{n-1}) \operatorname{Re}(e^{i\varphi} g_n) \quad \text{for } \varphi \in J.$$

3. Application to Bessel functions and Lommel polynomials. Let $J_\nu(z) = (z/2)^\nu \phi_\nu(z)$ be the ν th Bessel function, where

$$\phi_\nu(z) := \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{n! \Gamma(\nu + n + 1)}, \quad \nu, z \in \mathbb{C}, \nu \neq -1, -2, \dots$$

Then (see also [5, Chapter III])

$$(5) \quad \phi'_\nu(z) = -(z/2)\phi_{\nu+1}(z),$$

$$(6) \quad \phi_\nu(z) = (\nu + 1)\phi_{\nu+1}(z) - (z/2)^2 \phi_{\nu+2}(z),$$

$$(7) \quad z\phi''_\nu(z) + (2\nu + 1)\phi'_\nu(z) + z\phi_\nu(z) = 0.$$

Here (7) implies that all ϕ_ν have simple zeros. Then (5) shows that $\phi_\nu, \phi_{\nu+1}$ have no common zeros. Using [2, Theorem 5.16], we obtain from (6) that

$$(8) \quad \frac{\phi_{\nu+1}(z)}{\phi_\nu(z)} = \frac{1}{\nu + 1} - \frac{(z/2)^2}{\nu + 2} - \frac{(z/2)^2}{\nu + 3} - \dots,$$

where the infinite continued fraction is uniformly convergent on each compact set of $\mathbb{C}$ avoiding the poles of $\phi_{\nu+1}/\phi_\nu$ —i.e., the zeros of ϕ_ν . After applying an equivalence transformation to (8), we obtain

$$(9) \quad \frac{(\nu + 1)\phi_{\nu+1}(z)}{\phi_\nu(z)} = \frac{1}{1} + \frac{c_1 c_2 z^2}{1} + \frac{c_2 c_3 z^2}{1} + \dots,$$

where $c_n = 1/2i(\nu + n)$, $n \geq 1$. Applying Theorem 2 to the sequence of approximants of (9) or, equivalently, of (8) yields

THEOREM 4. *If $\nu = \nu_1 + i\nu_2$, $\nu_1, \nu_2 \in \mathbb{R}$, such that $0 \leq \arg(\nu + 1) < \pi$, and if*

$$B_{n,\nu}(z) = (\nu + n)B_{n-1,\nu}(z) - (z/2)^2 B_{n-2,\nu}(z), \quad n \geq 1, \\ B_{0,\nu} = 1, \quad B_{-1,\nu} = 0,$$

then $B_{n,\nu}(z) \neq 0$ for $n \geq 1$ and

- (a) *all $z = |z|e^{i\vartheta}$ satisfying $\arg(\nu + 1) < \vartheta \leq \pi$ or $\arg(\nu + 1) + \pi < \vartheta \leq 2\pi$,*
- (b) *all $z \in \mathbb{C}$ satisfying $|\operatorname{Im} z| \leq \nu_2$, provided $\nu_2 > 0$.*

Also,

- (c) *if $0 \leq \arg(\nu + 1) < \pi/2$, then, in addition to (a) and (b), $B_{n,\nu}(z) \neq 0$ for $n \geq 1$ and all $z \in \mathbb{C}$ satisfying $(\operatorname{Re} z)^2 \leq (\nu_1 + 1)(\nu_1 + 2)$.*

Because $B_{n,\bar{\nu}}(z) = \overline{B_{n,\nu}(\bar{z})}$, the zeros of $B_{n,\bar{\nu}}$ are the conjugates of the zeros of $B_{n,\nu}$.

PROOF. With the notations of Theorem 2 we obtain $e_n = 0$, $n \geq 1$, $\alpha = -\arg(\nu + 1) - \pi/2 \leq \psi_n < \beta = -\pi/2$ and $\beta - \alpha = \arg(\nu + 1)$. Then (a) follows from Theorem 2(a). Furthermore,

$$I = \{\varphi: -\pi \leq \varphi < -\arg(\nu + 1)\}$$

and

$$r(\varphi) = \inf_{n \geq 2} 4 \operatorname{Re}(e^{i\varphi} i(\nu + n - 1)) \operatorname{Re}(e^{i\varphi} i(\nu + n)) \quad \text{for } \varphi \in I.$$

Then Theorem 2(b) implies $B_{n,\nu}(z) \neq 0$ for $n \geq 1$ and all $z \in \bigcup_{\varphi \in I} S(r(\varphi), \varphi)$. Choosing especially $\varphi = -\pi$ and $\varphi = -\pi/2$ yields (b) and (c).

COROLLARY 1. If $\nu = \nu_1 + i\nu_2$, $\nu_1, \nu_2 \in \mathbf{R}$, such that $0 \leq \arg(\nu + 1) < \pi$, then $\phi_\nu(z) \neq 0$ for

- (a) all $z = |z|e^{i\vartheta}$ satisfying $\arg(\nu + 1) < \vartheta < \pi$ or $\arg(\nu + 1) + \pi < \vartheta < 2\pi$,
- (b) all $z \in \mathbf{C}$ satisfying $|\operatorname{Im} z| < \nu_2$, provided $\nu_2 > 0$.

Also,

(c) if $0 \leq \arg(\nu + 1) < \pi/2$, then, in addition to (a) and (b), $\phi_\nu(z) \neq 0$ for all $z \in \mathbf{C}$ satisfying $(\operatorname{Re} z)^2 < (\nu_1 + 1)(\nu_1 + 2)$. (See also [5, p. 486] for the special case $\nu + 1 > 0$.)

Because $\phi_{\bar{\nu}}(z) = \overline{\phi_\nu(\bar{z})}$, the zeros of $\phi_{\bar{\nu}}$ are the conjugates of the zeros of ϕ_ν .

REMARK. Because of the asymptotic formula [5, p. 199]

$$J_\nu(z) \sim (2/\pi z)^{1/2} \cos(z - \nu\pi/2 - \pi/4),$$

valid for $z \rightarrow \infty$, $-\pi < \arg z < \pi$, the “large” zeros of J_ν are near the lines $\operatorname{Im} z = \operatorname{Im} \nu\pi/2$ for $\operatorname{Re} z > 0$ or $\operatorname{Im} z = -\operatorname{Im} \nu\pi/2$ for $\operatorname{Re} z < 0$, provided $\operatorname{Im} \nu > 0$. See also [5, pp. 495–497].

Next, the Lommel polynomials $g_{m,\nu}(z)$, being related to $B_{m,\nu}(z)$, are defined as follows [5, p. 303]:

$$g_{m,\nu}(z) := \sum_{0 \leq n \leq m/2} (-1)^n \binom{m-n}{n} z^n \frac{\Gamma(\nu + m - n + 1)}{\Gamma(\nu + n + 1)},$$

$$\nu, z \in \mathbf{C}, \nu \neq -1, -2, \dots$$

Then

(10)

$$g_{m,\nu}(z) = (\nu + m)g_{m-1,\nu}(z) - zg_{m-2,\nu}(z), \quad m \geq 1, \quad g_{0,\nu} = 1, \quad g_{-1,\nu} = 0.$$

Therefore $g_{m,\nu}$ and $g_{m-1,\nu}$ have no common zeros. Furthermore by [5, p. 302], $\lim_{m \rightarrow \infty} g_{m,\nu}(z^2/4)/\Gamma(\nu + m + 1) = \phi_\nu(z)$ uniformly on each compact set in $\mathbf{C}$. (10) yields

$$\frac{g_{m-1,\nu}(z)}{g_{m,\nu}(z)} = \frac{1}{\nu + m} - \frac{z}{\nu + m - 1} - \dots - \frac{z}{\nu + 1}$$

or, after applying an equivalence transformation,

$$(11) \quad \frac{(\nu + m)g_{m-1,\nu}(z)}{g_{m,\nu}(z)} = \frac{1}{1} + \frac{c_1 c_2 z}{1} + \dots + \frac{c_{m-1} c_m z}{1},$$

where $c_n = 1/(\nu + m + 1 - n)$, $1 \leq n \leq m$.

Applying Theorem 1 to (11) yields

THEOREM 5. If $\nu = \nu_1 + i\nu_2$, $\nu_1, \nu_2 \in \mathbf{R}$, such that $0 \leq \arg(\nu + 1) < \pi$, then for each $m \in \mathbf{N}$, $g_{m,\nu}(z) \neq 0$ for

- (a) all $z = |z|e^{i\vartheta}$ satisfying $2\arg(\nu + 1) < \vartheta < 2\pi + 2\arg(\nu + m)$,
- (b) all $z \in \mathbf{C}$ satisfying $|z| \leq \operatorname{Re} z + \nu_2^2/2$.

Also,

(c) if $0 \leq \arg(\nu + 1) < \pi/2$, then, in addition to (a), (b), $g_{m,\nu}(z) \neq 0$ for all $z \in \mathbf{C}$ satisfying $|z| \leq \frac{-\operatorname{Re} z + (\nu_1 + 1)(\nu_1 + 2)}{2}$.

Because $g_{m,\bar{\nu}}(z) = \overline{g_{m,\nu}(\bar{z})}$, the zeros of $g_{m,\bar{\nu}}$ are the conjugates of the zeros of $g_{m,\nu}$.

PROOF. With the notation of Theorem 1, we obtain, for $1 \leq n \leq m$, $\alpha = -\arg(\nu + 1) - \pi/2 \leq \psi_n \leq \beta = -\arg(\nu + m) - \pi/2$. Then (a) follows from Theorem 1(a). Furthermore,

$$I = \{ \varphi: -\arg(\nu + m) - \pi < \varphi < -\arg(\nu + 1) \}$$

and

$$r(\varphi) = \min_{2 \leq n \leq m} \operatorname{Re}(e^{i\varphi} i(\nu + n - 1)) \operatorname{Re}(e^{i\varphi} i(\nu + n)) \quad \text{for } \varphi \in I.$$

Then Theorem 1(b) implies $g_{m,\nu}(z) \neq 0$ for all $z \in \bigcup_{\varphi \in I} P(r(\varphi), \varphi)$. Choosing especially $\varphi = -\pi$ and $\varphi = -\pi/2$ yields (b) and (c).

REMARK. Using (9) and applying Theorem 2 to the infinite continued fraction (with arbitrary $c_\nu, e_\nu \in \mathbb{C}, c_\nu \neq 0, 1 \leq \nu \leq n$)

$$\begin{aligned} & \frac{1}{1 + e_1 z} + \frac{c_1 c_2 z^2}{1 + e_2 z} + \cdots + \frac{c_{n-1} c_n z^2}{1 + e_n z + c_n c_{n+1} z^2 (\nu + 1) \phi_{\nu+1}(z) / \phi_\nu(z)} \\ &= \frac{(\nu + 1) c_n c_{n+1} z^2 A_{n-1}(z) \phi_{\nu+1}(z) + A_n(z) \phi_\nu(z)}{(\nu + 1) c_n c_{n+1} z^2 B_{n-1}(z) \phi_{\nu+1}(z) + B_n(z) \phi_\nu(z)}, \end{aligned}$$

one can obtain information on the location of zeros of

$$(\nu + 1) c_n c_{n+1} z^2 B_{n-1}(z) \phi_{\nu+1}(z) + B_n(z) \phi_\nu(z)$$

or, using (5), $B_n(z) \phi_\nu(z) - 2(\nu + 1) c_n c_{n+1} z B_{n-1}(z) \phi'_\nu(z)$.

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