HIGHER ORDER SINGULARITIES OF MORPHISMS TO PROJECTIVE SPACE

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ABSTRACT. This paper proves existence theorems for higher order singularities of a finite morphism to $\mathbf{P}^n$ and deduces a result on simple connectivity of varieties admitting a finite morphism of bounded singularity.

The singularities are obtained by successive degeneration of double points of f . Our main tool is R. Schwarzenberger's notion of generalized secant sheaves and the connectedness theorem obtained by W. Fulton and the author.

Conventions. Throughout, varieties will be complete—possibly reducible and singular, unless mentioned otherwise—and defined over an algebraically closed field k of arbitrary characteristic.

When we say that a variety is connected, it includes the assertion that it is nonempty.

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1. Higher order singularities. For a finite morphism $f: X \rightarrow Y$ of irreducible varieties, we say that a closed point $x \in X$ is a q th order singularity if

$$\dim_k(\mathcal{O}_{X,x}/f^*m_{f(x)} + m_x^{q+1}) \geq q + 1,$$

where $\mathcal{O}_{X,x}$ is the local ring of X at x and m_x (resp. $m_{f(x)}$) is the maximal ideal at x (resp. $m_{f(x)}$). In this notion every point of X is a 0th order singularity, and a first order singularity is a ramification point of f . Let $S^q = S^q(f)$ denote the set of q th order singularity points.

If $\mathcal{P}_{X/Y}^q$ denotes the coherent $\mathcal{O}_X$ -algebra of q th order relative principal parts, there is an isomorphism

$$\mathcal{P}_{X/Y}^q \otimes_{\mathcal{O}_X} k(x) \simeq \mathcal{O}_{X,x}/f^*m_{f(x)} + m_x^{q+1}$$

for any closed point x of X [EGA, IV.16.4.5 and 16.4.12]. Therefore

$$S^q = \{x \in X \mid \dim_k(\mathcal{P}_{X/Y}^q(x)) \geq q + 1\}$$

constitutes the q th order singularity locus. By semicontinuity of the dimension of the fibres of a coherent sheaf, S^q is a closed subscheme of X .

In the following, let

$$\sigma_q = \dim(X) - q(\dim(Y) - \dim(X) + 1),$$

the expected dimension of S^q .

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In fact, if Y is nonsingular, then every irreducible component of S^q , containing a smooth point x of X , at which the map on Zariski tangent spaces

$$(df)_x: T_{X,x} \rightarrow T_{Y,f(x)}$$

has rank $\geq \dim(X) - 1$, has dimension $\geq \sigma_q$ [R, 3.8].

Smooth points of X in S^q , where the map on Zariski tangent spaces $(df)_x: T_{X,x} \rightarrow T_{Y,f(x)}$ has rank $\geq \dim(X) - 1$, relate to the Thom-Boardman singularities $S_{1,\dots,1}$ (q indices) of a map of differentiable manifolds [T]. Namely assume S^{q-1} is smooth at x of dimension σ_{q-1} . Then the map on Zariski tangent spaces

$$d(f|_{S^{q-1}})_x: T_{S^{q-1},x} \rightarrow T_{Y,f(x)}$$

has rank $= \dim(S^{q-1}) - 1$ if and only if $x \in S^q$ ($\text{char } k = 0$) [R, 11.2].

2. Statements. Our existence theorem derives from the existence of degeneration of double points of f at a point in S^{q-1} .

2.1. THEOREM. *Let X be an irreducible variety and $f: X \rightarrow \mathbf{P}^m$ a finite morphism. Fix q with $\sigma_q \geq 0$. Assume S^{q-1} has an irreducible component V of dimension $\geq \sigma_{q-1}$, and the restriction of $f: f^{-1}(f(V)) \rightarrow f(V)$ is not one-to-one. Then S^q is nonempty and has an irreducible component of dimension $\geq \sigma_q$.*

2.2. COROLLARY. *Let X be a normal and irreducible variety. If X admits a finite morphism $f: X \rightarrow \mathbf{P}^m$ with $S^q = \emptyset$ for some q with $\sigma_q \geq 0$, then the fundamental group of X is trivial.*

The above assertions are proved in §4; an easy induction argument gives the following corollary.

2.3. COROLLARY. *Let X be an irreducible variety and $f: X \rightarrow \mathbf{P}^m$ a finite morphism with $\#f^{-1}(f(x)) \geq 2$ for all $x \in X$. Then S^q is nonempty and has an irreducible component of dimension $\geq \sigma_q$ for all q with $\sigma_q \geq 0$.*

T. Gaffney [G] has obtained similar results with similar methods, using another definition of multiplicity.

For coverings of $\mathbf{P}^m$, Gaffney and R. Lazarsfeld obtained remarkable results [G-L]. Their main assertions are that there exist points with local degree $\geq \min(d, m+1)$, where d is the degree of the covering; and that a normal variety X , representable as a cover of $\mathbf{P}^m$ with degree $\leq m$, is algebraically simply connected.

P. Deligne [D] and W. Fulton (cf. [F-L]) show, using the topological version of the connectedness theorem for projective space, that X is in fact topologically simply connected.

For a flat and generically étale covering, the local degree of [G-L] equals $\dim_k(\mathcal{O}_{X,x}/f^*m_{f(x)})$. Clearly

$$\dim_k(\mathcal{O}_{X,x}/f^*m_{f(x)}) \geq \dim_k(\mathcal{O}_{X,x}/f^*m_{f(x)} + m_x^{q+1}),$$

and their existence assertion and simple-connectivity result are in accordance with 2.1 and 2.2 for coverings.

It should be observed that some assumption, such as normality, on X is necessary in the π_1 -result for coverings, as shown by an example of A. Landman.

J. Roberts [R] studied higher order singularities of generic linear projections $X \rightarrow \mathbf{P}^m$.

3. Degeneration and generalized secant sheaves. In this section, we establish that degeneration of double points of f at a point in S^{q-1} gives rise to a point in S^q . Our main tool in proving this is Schwarzenberger's notion of generalized secant sheaves [Sc] and a result of A. Kato [K] relating these to the sheaves of relative principal parts.

3.1. PROPOSITION. *Let $f: X \rightarrow Y$ be a finite, projective morphism between irreducible varieties, and let V be an irreducible component of $S^{q-1}(f)$. If $Z \neq \Delta_V$ is an irreducible component of $V \times_Y X$, then*

$$\delta^{-1}(Z \cap \Delta_V) \subset S^q(f),$$

where Δ_V is the image of the diagonal morphism $\delta: V \rightarrow V \times_Y X$.

Let $f: X \rightarrow Y$ be a finite, projective morphism, and let $X_n = X \times_Y \cdots \times_Y X$ denote the n -fold relative product of X over Y .

Consider morphisms $h_t: X_n \rightarrow X_n \times_Y X$, where

$$h_t: (x_1, \dots, x_n) \rightarrow (x_1, \dots, x_n) \times x_t.$$

If W_n (resp. $\tilde{W}_n$) denotes the union of the images of h_t , $t = 1, \dots, n$ (resp. $t = 1, \dots, n-1$), then the two diagrams below constitute product schemes in the sense of [Sc]:

$$\begin{array}{ccc} & & X_n \\ & \nearrow h & \nearrow \pi_1 \\ W_n & \longrightarrow & X_n \times_Y X \\ & \searrow g & \searrow \pi_2 \\ & & X \end{array}, \quad \begin{array}{ccc} & & X_n \\ & \nearrow \tilde{h} & \nearrow \pi_1 \\ \tilde{W}_n & \longrightarrow & X_n \times_Y X \\ & \searrow \tilde{g} & \searrow \pi_2 \\ & & X \end{array}$$

Let $\Sigma^n = h_* g^*(\mathcal{O}_X)$, $\tilde{\Sigma}^n = \tilde{h}_* \tilde{g}^*(\mathcal{O}_X)$ denote the associated generalized secant sheaves which are coherent [EGA, III.2.2.1].

3.2. LEMMA. *With notation as above, the following holds:*

- (i) $\pi_1^*(\Sigma^n) \simeq \tilde{\Sigma}^{n+1}$, where π_1 is the projection $\pi_1: X_{n+1} = X_n \times_Y X \rightarrow X_n$,
- (ii) If $(x_1, \dots, x_n, x_{n+1}) \in W_n \setminus \tilde{W}_n$, then $\dim_{k(x)}(\Sigma^n(x)) > \dim_{k(x)}(\tilde{\Sigma}(x))$, where $x = (x_1, \dots, x_n) \in X_n$.

PROOF. A. Kato [K, p.103] establishes (i). To obtain (ii), let $\mathcal{J}$ be the ideal defined by

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_{W_n} \rightarrow \mathcal{O}_{\tilde{W}_n} \rightarrow 0.$$

Since h is finite, we need only show that $\mathcal{J}_{(x_1, \dots, x_{n+1})} \neq 0$, which follows from our assumption.

3.3. LEMMA (A. KATO [K, 3.3]). *If $\Delta: X \rightarrow X_{n+1}$ is the diagonal morphism, then $\Delta^*(\Sigma^{n+1}) \simeq \mathcal{P}_{X/Y}^n$.*

PROOF OF 3.1. Adopting notation as in the proposition, let $z \in \delta^{-1}(Z \cap \Delta_V)$. We can find a curve T and a morphism $c: T \rightarrow Z$, with $c(t_0) = (z, z)$ for some $t_0 \in T$ and $c(t) \notin \Delta_V$ for $t \in T \setminus \{t_0\}$. Projecting onto the factors of $V \times_Y X$ yields morphisms $c_1, c_2: T \rightarrow X$.

Consider

$$\tilde{c}(t) = (c_1(t), \dots, c_1(t), c_2(t)) \in X_{q+1},$$

with $\tilde{c}(t_0) = (z, \dots, z)$. For $t \neq t_0$,

$$\tilde{c}(t) \times c_1(t) \in W_{q+1} \setminus \tilde{W}_{q+1}.$$

Hence by 3.2(ii)

$$\dim_{k(\tilde{c}(t))}(\Sigma^{q+1}(\tilde{c}(t))) > \dim_{k(\tilde{c}(t))}(\tilde{\Sigma}^{q+1}(\tilde{c}(t))).$$

By 3.2(i), 3.3 and assumption on V , we get that

$$\dim_{k(\tilde{c}(t))}(\Sigma^{q+1}(\tilde{c}(t))) \geq q + 1$$

for $t \neq t_0$. Finally we conclude by semicontinuity of the dimension of the fibres of a coherent sheaf, and by 3.3, that

$$\dim_{k(z)}(\mathcal{P}_{X/Y}^q(z)) \geq q + 1.$$

4. Proofs of main results.

PROOF OF 2.1. Let $f: X \rightarrow \mathbf{P}^m$ be a finite morphism and V an irreducible component of S^{q-1} with $\dim(V) \geq \sigma_{q-1}$. Consider the finite morphism

$$F = f|_V \times f: V \times X \rightarrow \mathbf{P}^m \times \mathbf{P}^m,$$

where $V \times X$ is irreducible and $\dim(V \times X) \geq \sigma_{q-1} + \dim(X)$. By 5.5(i), $F^{-1}(\Delta_{\mathbf{P}^m})$ is $(\sigma_{q-1} + \dim(X) - m - 1)$ -connected. By assumption Δ_V is not the only irreducible component of $F^{-1}(\Delta_{\mathbf{P}^m})$, so there must be an irreducible component $Z \neq \Delta_V$ of $F^{-1}(\Delta_{\mathbf{P}^m})$ with

$$\dim(Z \cap \Delta_V) \geq \sigma_{q-1} + \dim(X) - m - 1,$$

and we are finished by 3.1, because the right side above is equal to σ_q .

PROOF OF 2.2. For $f: X \rightarrow \mathbf{P}^m$ as in the corollary, let q be maximal so that S^q has an irreducible component V of dimension $\geq \sigma_q$. By assumption, $\sigma_{q+1} \geq 0$.

Let $\pi: V^* \rightarrow V$ be the normalization of V and consider

$$(f \circ \pi) \times f: V^* \times X \rightarrow \mathbf{P}^m \times \mathbf{P}^m.$$

Our main theorem (2.1) tells us—by maximality of q —that the restriction of $f: f^{-1}(f(V)) \rightarrow f(V)$ is one-to-one, and therefore

$$((f \circ \pi) \times f)^{-1}(\Delta_{\mathbf{P}^m}) \simeq V^*.$$

The topological version of the connectedness theorem for projective space [D] (cf. [F-L]) now implies that the homomorphism $\pi_1(V^*) \rightarrow \pi_1(V^* \times X)$ is a surjection, which forces $\pi_1(X)$ to be trivial.

5. An l -connectedness theorem for projective space.

5.1. DEFINITION. Fix $l \geq 0$. A variety X is said to be l -connected if all its irreducible components have dimension greater or equal to $l + 1$, and for all its closed subvarieties V , with $\dim(V) < l$, $X \setminus V$ is connected.

With this definition, 0-connected means that all irreducible components of X are at least curves and X is connected. It is important to note that an irreducible variety X of dimension ≥ 1 is $(\dim(X) - 1)$ -connected but *not* $\dim(X)$ -connected.

The following definition and lemma provide another useful and perhaps more intuitive way of viewing this notion.

5.2. DEFINITION. A sequence $Z_0, \dots, Z_n$ of closed, irreducible subvarieties of X , all of dimension greater than or equal to $l + 1$, with $\dim(Z_{i-1} \cap Z_i) \geq l$ for all $i = 1, \dots, n$, will be called an l -join within X and Z_0, Z_n are referred to as the extreme components.

5.3. LEMMA. A variety X is l -connected if and only if any two irreducible components of X are the extreme components of an l -join within X . Equivalently, $X = Z_0 \cup \dots \cup Z_n$ for some l -join $Z_0, \dots, Z_n$.

PROOF. Assume $X = Z_0 \cup \dots \cup Z_n$, where $Z_0, \dots, Z_n$ is an l -join within X , and let V be a closed subvariety of X with $\dim(V) < l$. Then $X \setminus V = (Z_0 \setminus V) \cup \dots \cup (Z_n \setminus V)$, and $Z_i \setminus V$ are irreducible and nonempty for all $i = 0, \dots, n$ by reason of dimension. Likewise, $(Z_{i-1} \setminus V) \cap (Z_i \setminus V) = (Z_{i-1} \cap Z_i) \setminus V$ are nonempty for all $i = 1, \dots, n$. We conclude that $X \setminus V$ is connected.

Next assume that X is l -connected. Let X' be the union of all irreducible components of X , which can be l -joined to a given irreducible component Z' , and let X'' be the union of the other irreducible components. By definition of X' and X'' , $\dim(X' \cap X'') < l$; hence $X \setminus (X' \cap X'')$ is connected; i.e., $X' = X$.

5.4. LEMMA. Let $X_0, \dots, X_s$ be l -connected varieties, and assume $\dim(X_{i-1} \cap X_i) \geq l$ for all $i = 1, \dots, s$. Then the union $X_0 \cup \dots \cup X_s$ is l -connected.

PROOF. By assumption we can find irreducible components Z_{i-1} and Z'_i of X_{i-1} and X_i such that $\dim(Z_{i-1} \cap Z'_i) \geq l$ for all $i = 1, \dots, s$. Since X_i is l -connected, 5.3 allows us to display X_i as an l -join, which we, without restriction, can assume has Z'_i and Z_i among its extreme components. The union of all the l -joins then constitutes an l -join exhausting X , which therefore by 1.3 must be l -connected.

5.5. THEOREM. Let $P = \mathbf{P}^m \times \dots \times \mathbf{P}^m$ be a product of r copies of projective m -space, and let Δ be the image of the diagonal embedding of $\mathbf{P}^m$ in P . Let $d = \text{codim}(\Delta, P) = (r - 1)m$. Assume X is complete and l -connected, $l \geq d$, and consider a morphism $f: X \rightarrow P$.

- (i) If f is finite, then $f^{-1}(\Delta)$ is $(l - d)$ -connected.
- (ii) If for all $p \in P \setminus \Delta$, $\dim(f^{-1}(p)) \leq l - d$, then $f^{-1}(\Delta)$ is 0-connected.

Before proceeding to the proof, recall the following properties enjoyed by projective space, of which (ii) is the connectedness theorem of [F-H] and (i) is another way of saying that subvarieties of complementary dimension in projective space must meet.

5.6. PROPOSITION. *Let $P = \mathbf{P}^m \times \cdots \times \mathbf{P}^m$ be the product of r copies of projective m -space, and let Δ be the image of the diagonal embedding of $\mathbf{P}^m$ in P . For any morphism $f: X \rightarrow P$ from an irreducible and complete variety X :*

- (i) $f^{-1}(\Delta) \neq \emptyset$ if $\text{codim}(f(X), P) \leq m$.
- (ii) $f^{-1}(\Delta)$ is connected if $\text{codim}(f(X), P) < m$.

PROOF OF 5.5. (i) To begin with, assume that X is irreducible of dimension $\geq l + 1$, $l \geq d$, and let $V \subset f^{-1}(\Delta)$ be a closed subvariety, such that $f^{-1}(\Delta) \setminus V$ is disconnected. We must show that $\dim(V) \geq l - d$. Assuming this is not the case, we can choose a linear subspace $L \subset \Delta$ of codimension $l - d$, such that $V \cap f^{-1}(L) = \emptyset$. The image of every irreducible component of $f^{-1}(\Delta)$ has dimension $\geq l - d + 1$; hence $f^{-1}(L)$ meets every irreducible component of $f^{-1}(\Delta)$. Therefore $f^{-1}(L)$ is disconnected. On the other hand, consider

$$f \times i: X \times L \rightarrow \tilde{P} = P \times \mathbf{P}^m,$$

where $i: L \rightarrow \Delta \simeq \mathbf{P}^m$ is the inclusion. Then

$$\begin{aligned} \text{codim}((f \times i)(X \times L), \tilde{P}) &= \text{codim}(f(X), P) + l - d \\ &= \dim(P) - (l + 1) + l - d < m; \end{aligned}$$

therefore by 5.6(ii), $f^{-1}(L) \simeq (f \times i)^{-1}(\tilde{\Delta})$ is connected, contradicting our assumption.

In the general case, let $Z_0, \dots, Z_n$ be an l -join with $X = Z_0 \cup \cdots \cup Z_n$. Then $f^{-1}(\Delta) = W_0 \cup \cdots \cup W_n$, $W_i = (f|_{Z_i})^{-1}(\Delta)$ is $(l - d)$ -connected by what we have just seen. By assumption $\dim(Z_{i-1} \cap Z_i) \geq l$ for all $i = 1, \dots, n$; hence $W_{i-1} \cap W_i$ is nonempty and of dimension $\geq l - d$ for all $i = 1, \dots, n$ by 5.6(i). Therefore we conclude by 5.4 that $f^{-1}(\Delta)$ is $(l - d)$ -connected.

(ii) Our assumptions on the fiber dimensions imply that an irreducible component Z of X is either mapped to Δ , or $\text{codim}(f(Z), P) < m$. In either case $(f|_Z)^{-1}(\Delta)$ is 0-connected, the latter case following from 5.6(ii), and the former being trivial. Now if $X = Z_0 \cup \cdots \cup Z_n$ is an l -join, then for every $i = 1, \dots, n$, either $f(Z_{i-1} \cap Z_i) \subset \Delta$ or $\text{codim}(f(Z_{i-1} \cap Z_i), P) \leq m$. In both cases we conclude that $f(Z_{i-1} \cap Z_i) \cap \Delta \neq \emptyset$; the first case is trivial, and the second follows from 5.6(i). Our assertion now follows from 5.4, because $f^{-1}(\Delta) = W_0 \cup \cdots \cup W_n$, where $W_i = (f|_{Z_i})^{-1}(\Delta)$ is 0-connected and $W_{i-1} \cap W_i \neq \emptyset$ for all $i = 1, \dots, n$.

REFERENCES

- [D] P. Deligne, *Le groupe fondamental du complément d'une courbe plane n'ayant des points doubles ordinaires est abélien*, Séminaire Bourbaki, 1979/80, no. 543.
- [EGA] A. Grothendieck and J. Dieudonné, *Eléments de géométrie algébrique*. II, III, IV, Inst. Hautes Etude Sci. Publ. Math. **11** (1961), **32** (1967).
- [F-H] W. Fulton and J. Hansen, *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings*, Ann. of Math. (2) **110** (1979), 159–166.
- [F-L] W. Fulton and R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, Lecture Notes in Math., vol. 862, Springer-Verlag, Berlin and New York, 1981, pp. 26–93.
- [G] T. Gaffney, *Multiple points and associate ramification loci, singularities*, Proc. Sympos. Pure Math. vol. 40, parts 1, 2, Amer. Math. Soc., Providence, R.I., 1983, pp. 429–437.

[G-L] T. Gaffney and R. Lazarsfeld, *On the ramification of branched coverings of $\mathbf{P}^m$* , Invent. Math. **59** (1980), 53–58.

[K] A. Kato, *Singularities of projective embedding (points of order n on an elliptic curve)*, Nagoya Math. J. **45** (1971), 97–107.

[L] R. Lazarsfeld, Thesis, Brown University.

[R] J. Roberts, *Singularity subschemes and generic projections*, Trans. Amer. Math. Soc. **212** (1975), 229–268.

[Sc] R. L. E. Schwarzenberger, *The secant bundle of a projective variety*, Proc. London Math. Soc. (3) **14** (1964), 369–384.

[T] R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier (Grenoble) **6** (1955/56), 43–87.

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