BEST L_1 -APPROXIMATION OF L_1 -APPROXIMATELY CONTINUOUS FUNCTIONS ON $(0, 1)^n$ BY NONDECREASING FUNCTIONS

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ABSTRACT. For $n \geq 1$, let Ω denote the open unit n -cube, $(0, 1)^n$. Let μ denote Lebesgue measure, let Σ consist of the Lebesgue measurable subsets of Ω , and let $L_1 = L_1(\Omega, \Sigma, \mu)$. Let A_1 consist of the approximately continuous functions in L_1 and let M consist of the equivalence classes in L_1 which contain nondecreasing functions. Let $f \in A_1$. It is shown that there is a unique best L_1 -approximation $[g]$ to f in M . If $n = 1$, g is continuous, but if $n > 1$, $[g]$ may not consist of a continuous function.

The case of bounded, approximately continuous functions f on the closed interval $[0, 1]$ is considered in [1]; continuous functions f on the closed n -cube $[0, 1]^n$, $n \geq 1$, are considered in [2]. In each of these cases there is a unique best L_1 -approximation f_1 by nondecreasing functions and f_1 is continuous.

The measure of the boundary of a cube is zero, and the following simple example ($n = 1$) shows that we cannot expect to get best nondecreasing L_1 -approximations unless we remove the boundary. Put $f(x) = (1 - x)^{-(1/2)}$, $0 \leq x < 1$, $f(1) = 1$.

When $f \in A_1$, the class $[f]$ in L_1 that contains f is comprised of f alone. There is a one-to-one correspondence between the integrable, strictly right continuous, nondecreasing functions g on Ω and the classes in L_1 which contain a nondecreasing function, so we identify M with strictly right continuous functions.

Let S be a measurable set in Ω . The upper metric density of S at a point p is

$$\limsup_{n \rightarrow \infty} \sup_Q \left[\frac{m(S \cap Q \cap \Omega)}{m(Q \cap \Omega)} : m(Q) < \frac{1}{n} \right],$$

where Q is an open n -cube containing p ; the lower metric density is defined similarly. If they are equal, the common value is called the metric density of S at p .

A function from Ω to R is said to be approximately continuous at a point p if, for every open set G containing $f(p)$, the set $f^{-1}(G)$ has metric density 1 at p . A function f is said to be approximately continuous on Ω if it is approximately continuous at every point of Ω .

For $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \Omega$, $x \leq y$ means $x_i \leq y_i$, $i \leq n$, and $x < y$ means $x_i < y_i$, $i \leq n$. A function $g: \Omega \rightarrow R$ is nondecreasing on Ω if $x \leq y$ implies that $g(x) \leq g(y)$; g is strictly right continuous if $g(x) = \inf_{y > x} g(y)$, $x \in \Omega$.

Consider a nondecreasing function g on Ω . For $u \in R^{n-1}$ and $t \in R$, put $p(u, t) = (u, 0) + t1_n$, $1_n = (1, \dots, 1) \in R^n$. Then put $g_u(t) = g(p(u, t))$ when

Received by the editors April 5, 1985. Presented at the 89th Summer Meeting of the A.M.S., Special Session on Numerical Approximation and Applications, Laramie, Tuesday, August 13, 1985.

1980 *Mathematics Subject Classification.* Primary 26B35, 41A52.

Key words and phrases. Approximate continuity, best approximation, L_1 , nondecreasing functions.

$p(u, t) \in \Omega$. For each u, g_u is a nondecreasing function defined on a bounded, open, connected subset (perhaps empty) of R . Consequently, g_u has a countable set of discontinuities. Moreover, if $p(u, t) \in \Omega$ and g_u is continuous at t , then g is continuous at $p(u, t)$. Fubini's theorem permits us to assert that g is continuous a.e. on Ω .

Let $f \in A_1$ and let M_f denote the set of all best L_1 -approximations to f in M . Lemma 3 in [4] shows that M_f contains a least element $\underline{f}$ and a greatest element $\bar{f}$. The proof of (8) in [4] assures us that $\mu([\underline{f} < f < \bar{f}]) = 0$.

THEOREM. *Let $f \in A_1$. Then there exists a unique best L_1 -approximation f_1 to f in M .*

PROOF. It suffices to show that $\underline{f} = \bar{f}$ or, recalling that $\underline{f}$ and $\bar{f}$ are taken to be strictly right continuous, equivalently that $\mu([\underline{f} < \bar{f}]) = 0$. To this end, let Ω_0 denote the set of points in Ω at which each of $\underline{f}$ and $\bar{f}$ is continuous. Since $\mu(\Omega - \Omega_0) = 0$, it suffices to show that $\underline{f} = \bar{f}$ on Ω_0 . Suppose, on the contrary, that $x_0 \in \Omega_0$ with $\underline{f}(x_0) < \bar{f}(x_0)$. Observe that, because f is approximately continuous, $\{x: \underline{f}(x) < f(x) < \bar{f}(x)\} \subset (\Omega - \Omega_0)$. Hence, either $f(x_0) \geq \bar{f}(x_0)$ or $f(x_0) \leq \underline{f}(x_0)$. Symmetry permits us to consider only the first case. So we show that the supposition $f(x_0) \geq \bar{f}(x_0)$ leads to a contradiction: a better L_1 -approximation $g \in M_f$. Define g on Ω by

$$g(x) = \max\{(\underline{f}(x_0) + \bar{f}(x_0))/2, \underline{f}(x_0)\}, \quad x > x_0,$$

and $g(x) = \underline{f}(x)$, otherwise. Put $J = [g > \underline{f}]$; $\mu(J) > 0$. If $f \geq g$ on J , then

$$\int_{\Omega} |f - \underline{f}| - \int_{\Omega} |f - g| = \int_J |f - \underline{f}| - \int_J |f - g| = \int_J (g - \underline{f}) > 0;$$

this contradiction would establish the theorem. So we verify that $f \geq g$ on J by supposing that there exists $x_1 \in J$ with $f(x_1) < g(x_1)$ and obtaining another contradiction. To this end, observe that, because f_1 is approximately continuous, there exist x_2 and x_3 with $x_0 < x_2 < x_3 < x_1$, $f(x_2) > g(x_0)$, and $f(x_3) < g(x_1) = g(x_0)$. Put $x_4 = (x_0 + x_2)/2$ and $x_5 = (x_1 + x_3)/2$. Then (cf. [3 or 5]), $f(\{x: x_4 < x < x_5\})$ is connected, so there exists x_t with $x_0 < x_t < x_1$ and $g(x_t) = g(x_0)$. Put $\varepsilon = (g(x_1) - \underline{f}(x_1))/2$ and let $\delta > 0$ verify $B_{\infty}(x_t, \delta) = \{y \in R^n: \|y - x_t\|_{\infty} \leq \delta\} \subset \{x: x_0 < x < x_1\} \subset J$. Since f is approximately continuous at x_t and $\mu(\Omega - \Omega_0) = 0$, there exists $y \in B_{\infty}(x_t, \delta) \cap \Omega_0$ with $|f(y) - f(x_t)| < \varepsilon$. Thus, $y \in \Omega_0$ and $\underline{f}(y) < f(y) < \bar{f}(y)$; this contradiction completes a proof of the theorem.

When $n = 1$, the proof of Lemma 1 in [1] applies to show that f_1 is continuous on Ω . We conclude with an example to illustrate the fact that f_1 need not be continuous when $n > 1$.

EXAMPLE. For notational simplicity we consider the case $n = 2$ and $\Omega = (-1/2, 1/2)^2$. Define f on Ω by

$$\begin{aligned} f(x, y) &= 0, & x \leq 0, y \geq 0, \\ f(x, y) &= 0, & x \leq -y^2, y < 0, \\ f(x, y) &= 0, & (x - 2^{-5})^2 + y^2 \leq 2^{-10}, \\ f(x, y) &= 1, & x > 0, (x - 2^{-4})^2 + y^2 \geq 2^{-8} \text{ and } x \geq y^2 \text{ when } y > 0, \\ f(x, y) &= x/y^2, & y > 0, 0 < x < y^2, \\ f(x, y) &= (y^2 + x)/y^2, & y < 0, -y^2 < x \leq 0, \end{aligned}$$

$$f(\rho \cos \theta, \rho \sin \theta) = (\rho - 2^{-4} \cos \theta) / (2^{-4} \cos \theta), \\ -\pi/2 < \theta < \pi/2 \text{ and } 2^{-4} \cos \theta < \rho < 2^{-3} \cos \theta.$$

Then f is approximately continuous at zero and continuous elsewhere. But, $f_1(x, y) = 0$ if $x < 0$ and $f_1(x, y) = 1$ when $x \geq 0$.

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