EXACT SEQUENCES OF SPECTRA AND DUALITY

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ABSTRACT. We note that duality arguments show that, under certain circumstances, an exact sequence of spectra remains exact after smashing with a finite complex. Among various applications, we show that the validity for S^0 of G. W. Whitehead's symmetric products conjecture implies that the conjecture is "almost" true for all finite complexes.

1. Introduction. In [6], we studied sequences of spectra $\cdots \rightarrow E_2 \rightarrow E_1 \rightarrow E_0$ having the property that application of the 0th space functor Ω^∞ yields a sequence of spaces of the form $\cdots \rightarrow X_2 \times X_1 \rightarrow X_1 \times X_0 \rightarrow X_0$, with all maps being the obvious composites of projections and inclusions. Such an exact sequence induces a long exact sequence on homotopy groups in nonnegative degrees.

In this note, we observe that a simple duality argument shows that exact sequences often remain exact after being smashed with a finite complex. Thus there is more to be gleaned from knowing that a sequence is exact than might at first be apparent.

As applications of this principle, we offer the following three examples (which will be elaborated upon in §3).

In [4], D. Kahn and S. Priddy showed that, localized at a prime p , there is a stable map $\lambda: \Sigma^\infty B\Sigma_p \rightarrow \Sigma^\infty S^0$ such that $\Omega^\infty \lambda: Q_0 B\Sigma_p \rightarrow Q_0 S^0$ is the projection onto a direct factor. Here $QX = \Omega^\infty \Sigma^\infty X$, and $Q_0 S^0$ is the component of the basepoint. Thus $\pi_q^s(B\Sigma_p) \rightarrow \pi_q^s(S^0)$ is onto for $q > 0$. We use this to show that if Y is a finite complex and $n = 2 \dim Y + 1$, then $\Omega^n Q(B\Sigma_p \wedge Y) \rightarrow \Omega^n QY$ is also the projection onto a direct factor. Thus $\pi_q^s(B\Sigma_p \wedge Y) \rightarrow \pi_q^s(Y)$ is onto for $q > 2 \dim Y$ (this has been independently observed by J. Jones [3]).

In [5, 7], the Kahn-Priddy epimorphism was extended to a long exact sequence to conclude that, localized at p ,

$$(1.1) \quad \ker \{ \pi_q(\mathrm{SP}^{p^k} Y) \rightarrow \pi_q(\mathrm{SP}^{p^{k+1}} Y) \} = \ker \{ \pi_q(\mathrm{SP}^{p^k} Y) \rightarrow \pi_q(\mathrm{SP}^\infty Y) \}$$

is valid when $Y = S^0$. Here $\mathrm{SP}^m E$ denotes the m th symmetric product of a spectrum E . This was originally conjectured to be true for all Y by G. W. Whitehead [9], but a counterexample was found by P. Welcher [11] when $p = 2$, $k = 0$, $q = 5$ and $Y = S^0 \cup_p e^4$. In spite of this, we use the result for S^0 to show that if Y is a finite complex, then (1.1) is true for all but a finite number of exceptional pairs (k, q) .

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As a final example, recall that J. Becker showed that there is an infinite loop map $QBO(2) \rightarrow BO$ which is the projection onto a direct factor [2]. Using this, we show that there is a similar projection $\Omega^6 Q(BO(2) \wedge \mathbb{C}P^2) \rightarrow BU \times \mathbb{Z}$.

2. Exact sequences and duality. We recall some definitions from [6]. A spectrum P is *spacelike* if it is a wedge summand of a suspension spectrum. A fibration sequence $A \rightarrow B \xrightarrow{f} C$ is *short exact* if $\Omega^\infty f: \Omega^\infty B \rightarrow \Omega^\infty C$ is the projection onto a direct factor, so that $\Omega^\infty B = \Omega^\infty A \times \Omega^\infty C$. The map f is said to be *onto*.

A sequence of spectra $\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow E_0$ is *exact* if it is obtained from a diagram of the form

$$\begin{array}{ccccccc} \cdots & & E_2 & & E_1 & & E_0 \\ & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\ & X_2 & & X_1 & & X_0 & \end{array}$$

where each $E_{k+1} \rightarrow X_k \rightarrow E_k$ is short exact.

For the remainder of this paper, we will assume that all spectra are C. W. spectra of finite type (possibly localized at p). Thus any spectrum E considered will be the direct limit of a sequence of finite spectra.

The following is the key observation of this paper.

THEOREM 2.1. *Suppose that $\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow E_0$ is an exact sequence of spectra and that $\pi_q(X_k)$ is finite for $q \geq 0$ and $k \geq 1$. If Y is a finite complex such that the n -dual of Y is spacelike, then there is an exact sequence*

$$\cdots \rightarrow \Sigma^{-n} X_2 \wedge Y \rightarrow \Sigma^{-n} X_1 \wedge Y \rightarrow \Sigma^{-n} X_0 \wedge Y \rightarrow \Sigma^{-n} E_0 \wedge Y.$$

Note that, if Y is (-1) -connected, any $n \geq 2 \dim Y$ satisfies the conditions.

PROOF. To begin, it suffices to prove the theorem with the long exact sequence replaced by a short exact sequence $A \rightarrow B \rightarrow C$ such that $\pi_q(A)$ is finite for $q \geq 0$.

Standard arguments (see [6]) using the adjoint functors $(\Sigma^\infty, \Omega^\infty)$ show that $\Sigma^{-n} A \wedge Y \rightarrow \Sigma^{-n} B \wedge Y \rightarrow \Sigma^{-n} C \wedge Y$ will be exact if and only if

$$(2.2) \quad 0 \rightarrow [P, \Sigma^{-n} A \wedge Y] \rightarrow [P, \Sigma^{-n} B \wedge Y] \rightarrow [P, \Sigma^{-n} C \wedge Y] \rightarrow 0$$

is exact for all spacelike spectra P .

Recall the duality isomorphism $[X \wedge DY, Z] \cong [X, Z \wedge Y]$, valid if both X and Y are finite [1, p. 195]. Thus the exactness of $A \rightarrow B \rightarrow C$ implies that (2.2) is exact for all finite spacelike P , since $\Sigma^n P \wedge DY$ will be spacelike.

For the general case, $P = \varinjlim P_i$, where the P_i are finite spacelike spectra. There is a diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ \lim^1 [\Sigma P_i, \Sigma^{-n} A \wedge Y] & \rightarrow & \lim^1 [\Sigma P_i, \Sigma^{-n} B \wedge Y] & \rightarrow & \lim^1 [\Sigma P_i, \Sigma^{-n} C \wedge Y] & \rightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ [P, \Sigma^{-n} A \wedge Y] & \rightarrow & [P, \Sigma^{-n} B \wedge Y] & \rightarrow & [P, \Sigma^{-n} C \wedge Y] & & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow \lim [P_i, \Sigma^{-n} A \wedge Y] & \rightarrow & \lim [P_i, \Sigma^{-n} B \wedge Y] & \rightarrow & \lim [P_i, \Sigma^{-n} C \wedge Y] & \rightarrow & \lim^1 [P_i, \Sigma^{-n} A \wedge Y] \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

where the vertical sequences are Milnor exact sequences, and the top and bottom ones are part of standard six-term $\lim^1$ exact sequences. Under our assumption about $\pi_*(A)$, the sets $[\Sigma P_i, \Sigma^{-n}A \wedge Y]$ and $[P_i, \Sigma^{-n}A \wedge Y]$ are finite. The corresponding $\lim^1$ terms thus vanish, and the exactness of the middle horizontal sequence follows.

3. Examples. Our first applications of Theorem 2.1 are to the exact sequence

$$\cdots \rightarrow \Sigma L(2) \rightarrow \Sigma L(1) \rightarrow \Sigma L(0) \rightarrow \Sigma H\mathbf{Z}$$

constructed in our solution to Whitehead's conjecture for S^0 . Here all spectra are to be localized at p , $L(k) = \Sigma^{-k} \mathrm{SP}^{p^k} S^0 / \mathrm{SP}^{p^{k-1}} S^0$, and $H\mathbf{Z}$ is the integral Eilenberg-Mac Lane spectrum (which can be identified with $\mathrm{SP}^\infty S^0$ by the Dold-Thom Theorem). $L(0) = \Sigma^\infty S^0$ and $L(1) = \Sigma^\infty B\Sigma_p$, so exactness at $\Sigma L(0)$ is the Kahn-Priddy Theorem. (The extra suspension is a bit of a bonus.)

For $k \geq 1$, $L(k)$ has finite homotopy groups, since, for example, it is known to be a stable wedge summand of $B(\mathbf{Z}/p)^k$ [12]. Thus Theorem 2.1 can be directly applied to conclude that, for every finite complex Y , there is an exact sequence

$$(3.1) \quad \cdots \rightarrow \Sigma^{1-n} L(2) \wedge Y \rightarrow \Sigma^{1-n} L(1) \wedge Y \rightarrow \Sigma^{1-n} \wedge Y \rightarrow \Sigma^{1-n} H\mathbf{Z} \wedge Y$$

with n chosen so that the n -dual of Y is spacelike.

Consideration of the beginning of this sequence yields the next theorem.

THEOREM 3.2. *Let Y be a finite complex with n chosen as above. Let*

$$M = \max\{n - 1, \dim Y + 1\}.$$

- (a) $\pi_q^s(B\Sigma_p \wedge Y) \rightarrow \pi_q^s(Y)$ is onto for $q \geq M$.
- (b) Let $\alpha: \Sigma^d Y \rightarrow Y$ be any stable self-map. If $dN \geq M$ then the N th iterate of α lifts:

$$\begin{array}{ccc} & & B\Sigma_p \wedge Y \\ & \nearrow \text{dashed} & \downarrow \\ \Sigma^{dN} Y & \xrightarrow{\alpha^N} & Y \end{array}$$

- (c) $\Omega^M Q(B\Sigma_p \wedge Y) \rightarrow \Omega^M QY$ is the projection onto a direct factor.

All of these follow immediately from the exactness of (3.1) at $\Sigma^{1-n} Y$ together with the observation that $\Omega^M \Omega^\infty(H\mathbf{Z} \wedge Y) \simeq *$ since $M > \dim Y$.

Exactness of the whole sequence (3.1) can be used to attack Whitehead's conjectures.

THEOREM 3.3. *With Y and n as above, (1.1) is true unless $k + n - 1 > q \geq 2p^{k+1} - 2$.*

PROOF. The key is the observation that, for a spectrum E , $\mathrm{SP}^m E = \mathrm{SP}^m S^0 \wedge E$ [11]. Straightforward diagram chasing applied to (3.1) then gives the stated upper bound on q . The lower bound is trivial: connectivity arguments show that $\pi_q(\mathrm{SP}^{p^k} Y) \rightarrow \pi_q(\mathrm{SP}^\infty Y)$ is an isomorphism if $q < 2p^{k+1} - 2$.

EXAMPLES 3.4.

Y	n	(k, q) satisfying the inequalities of Theorem 3.3
$S^0 \cup_2 e^1$	2	$\emptyset$
$S^0 \cup_\eta e^2$	4	$(0, 2)$
$S^0 \cup_{\eta^2} e^3$	5	$(0, 2), (0, 3)$
$S^0 \cup_\nu e^4$	8	$(0, q), \quad 2 \leq q < 7,$
$S^0 \cup_\sigma e^8$	16	$(0, q), \quad 2 \leq q < 15,$ $(1, q), \quad 6 \leq q < 16,$ $(2, q), \quad 14 \leq q < 17$

When the attaching map is η or η^2 , easy diagram chasing eliminates the exceptional pairs (k, q) . (This includes Welcher's mod 2 examples [11, Corollary 6.5].)

EXAMPLE 3.5. Let p be odd and let $\alpha \in \pi_{2p-3}(S^0)$ be a generator. Then (1.1) is true for $Y = S^0 \cup_\alpha e^{2p-2}$. In this case, the inequalities leave open only the cases $(k, q) = (0, 2p - 2)$ or $(0, 2p - 1)$. Again it is easy to eliminate these cases.

For our next application of Theorem 2.1, we interpret Becker's result about BO as the construction of an onto map $f: \Sigma^\infty BO(2) \rightarrow ko\langle 0 \rangle$, where $ko\langle 0 \rangle$ is the 0-connected cover of the periodic real K -theory spectrum KO . Since the fiber of f has finite homotopy groups, Theorem 2.1 applies. Thus letting $Y = S^0 \cup_\eta e^2$, there is an onto map

$$(3.6) \quad \Sigma^{-4} \Sigma^\infty BO(2) \wedge Y \rightarrow \Sigma^{-4} ko\langle 0 \rangle \wedge Y.$$

Now note that $Y = \Sigma^{-2} CP^2$, $\pi_q(ko\langle 0 \rangle \wedge Y) \simeq \pi_q(KO \wedge Y)$ for $q > 2$, and $KO \wedge Y = KU = \Sigma^{-4} KU$ [1, p. 206]. These facts, together with (3.6), imply that there is an onto map $\Sigma^{-6} \Sigma^\infty BO(2) \wedge CP^2 \rightarrow KU$.

The next theorem follows.

THEOREM 3.7. *There is an infinite loop map $\Omega^6 Q(BO(2) \wedge CP^2) \rightarrow BU \times \mathbf{Z}$ which is a projection onto a direct summand.*

REMARK 3.8. In using Theorem 2.1, it occasionally happens that n can be chosen to be quite small. As an amusing example, let $Y = B(k)$, the k th mod 2 Brown-Gitler spectrum with $H^*(B(k)) = A/A\{\chi(\text{Sq}^i) | i > k\}$. It was conjectured by H. Miller [10] and proved by J. Lannes [8] that the $2k$ th dual of $B(k)$ is spacelike. Thus we conclude that a short exact sequence of spectra $A \rightarrow B \rightarrow C$ induces a short exact sequence

$$0 \rightarrow B(k)_q(A) \rightarrow B(k)_q(B) \rightarrow B(k)_q(C) \rightarrow 0$$

for $q \geq 2k$. Note that this range is essentially complementary to the range of q for which properties of $B(k)$ allow one to do computations.

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