A COUNTEREXAMPLE IN DYNAMICAL SYSTEMS OF THE INTERVAL

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ABSTRACT. In [1] it was proved that if the recurrent points of a continuous map of the unit interval form a closed set, then this map has no periodic point with period not equal to a power of 2, i.e. this map is of type 2^∞ . In this paper we will construct a continuous map of the interval which is of type 2^∞ and for which the set of recurrent points is not closed. By such a counterexample it may be shown that some of the results announced in [2] are not correct.

First, recall some elementary definitions.

Suppose that $f: I \rightarrow I$ is a continuous map from the interval $I = [0, 1]$ into itself. Let f^0 be the identity map on I . For any positive integer n define $f^n = f \circ f^{n-1}$.

A point $x \in I$ is said to be periodic if $x = f^m(x)$ for some $m > 0$. In this case $\min\{m > 0: f^m(x) = x\}$ is called the period of x , and the finite set $\{f^m(x): m = 1, 2, \dots\}$ is called the orbit of x .

A point $x \in I$ is said to be recurrent if for any neighborhood U of x , $f^m(x) \in U$ for some $m > 0$.

A subset S of I is said to be invariant if $f(S) \subset S$.

A continuous map of the interval is of type 2^∞ if it has no periodic point with period not equal to a power of 2.

We will define a map $\chi_\infty: I \rightarrow I$ having no periodic point with period not a power of 2 such that the set $R(\chi_\infty)$ of recurrent points is not closed.

1. Let $\xi_{[a,b],[c,d]}$ denote the linear map from $[a, b]$ to $[c, d]$ which carries a into c and b into d , and let $\tilde{\xi}_{[a,b],[c,d]}$ denote the linear map from $[a, b]$ to $[c, d]$ which carries a into d and b into c .

For any positive integer $k \geq 3$, let

$$a_k = \frac{1}{2^k}, \quad b_k = 1 - \frac{3}{2^k}, \quad c_k = 1 - \frac{2}{2^k} \quad \text{and} \quad d_k = 1 - \frac{1}{2^k},$$

and define a map $\mu_k: I \rightarrow I$ such that

$$\begin{aligned} \mu_k|_{[0, a_k]} &= \xi_{[0, a_k], [0, c_k]}, \\ \mu_k|_{[a_k, b_k]} &= \xi_{[a_k, b_k], [c_k, d_k]}, \\ \mu_k|_{[b_k, c_k]} &= \xi_{[b_k, c_k], [a_k, d_k]}, \\ \mu_k|_{[c_k, d_k]} &= \xi_{[c_k, d_k], [a_k, b_k]}, \\ \mu_k|_{[d_k, 1]} &= \xi_{[d_k, 1], [b_k, 1]}, \end{aligned}$$

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It is easy to see that the map μ_k is well defined and continuous.

The following proposition is obvious.

PROPOSITION 1. *The set of periodic points of μ_k is $\{0, 1, e_k\} \cup [a_k, b_k] \cup [c_k, d_k]$, where $e_k \in [b_k, c_k]$, in which $0, 1, e_k$ are fixed points of μ_k and every point in $[a_k, b_k] \cup [c_k, d_k]$ has period 2.*

Therefore, μ_k is of type 2^∞ .

2. Suppose $f: I \rightarrow I$ is a continuous map with fixed points 0 and 1. Define a map $\mu_k * f: I \rightarrow I$ such that

$$\mu_k * f|_{I-(a_k, b_k)} = \mu_k|_{I-(a_k, b_k)}$$

and

$$\mu_k * f|_{[a_k, b_k]} = \xi_{1, [c_k, d_k]} \circ f \circ \xi_{[a_k, b_k], I}.$$

It is easy to see that the map $\mu_k * f$ is well defined and continuous and has fixed points 0 and 1.

By definition,

$$\mu_k * f([a_k, b_k]) = [c_k, d_k] \quad \text{and} \quad \mu_k * f([c_k, d_k]) = [a_k, b_k].$$

Hence, we have

(a) in $[a_k, b_k] \cup [c_k, d_k]$ there is no periodic point of $\mu_k * f$ with odd period. Obviously, the points 0, 1, and e_k are fixed points of $\mu_k * f$. If $y \in (0, a_k)$, or if $y \in (d_k, 1)$, then y is not a periodic point of $\mu_k * f$ because there is some $m > 0$ such that $(\mu_k * f)^m(y) \in [a_k, d_k]$, and $[a_k, d_k]$ is an invariant subset of $\mu_k * f$. Similarly, if $y \in (b_k, c_k)$ and $y \neq e_k$, then y is not a periodic point of $\mu_k * f$ because there is some $m > 0$ such that $(\mu_k * f)^m(y) \in [a_k, b_k] \cup [c_k, d_k]$ and $[a_k, b_k] \cup [c_k, d_k]$ is an invariant subset of $\mu_k * f$. Thus,

(b) in $[0, a_k) \cup (b_k, c_k) \cup (d_k, 1]$ there are no periodic points of $\mu_k * f$ other than the fixed points 0, 1 and e_k .

In addition,

$$\begin{aligned} (\mu_k * f)^2|_{[a_k, b_k]} &= \mu_k * f|_{[c_k, d_k]} \circ \mu_k * f|_{[a_k, b_k]} \\ &= \mu_k|_{[c_k, d_k]} \circ \xi_{I, [c_k, d_k]} \circ f \circ \xi_{[a_k, b_k], I} \\ &= \xi_{[c_k, d_k], [a_k, b_k]} \circ \xi_{I, [c_k, d_k]} \circ f \circ \xi_{[a_k, b_k], I} \\ &= \xi_{I, [a_k, b_k]} \circ f \circ \xi_{[a_k, b_k], I}. \end{aligned}$$

From this, it follows that

(c) for any $m \geq 1$,

$$(\mu_k * f)^{2m}|_{[a_k, b_k]} = \xi_{I, [a_k, b_k]} \circ f^m \circ \xi_{[a_k, b_k], I}.$$

By statements (a), (b), and (c) one can prove the following proposition immediately.

PROPOSITION 2. (1) *If there is a periodic point of $\mu_k * f$ with period $n > 1$, then n is even and there is a periodic point of f with period $n/2$.*

Therefore, if f is of type 2^∞ , then $\mu_k * f$ is of type 2^∞ .

(2) *If x is a periodic point of f with period $m > 0$, then $y = \xi_{I, [a_k, b_k]}(x)$ is a periodic point of $\mu_k * f$ with period $2m$; furthermore, if for some $u \in I$*

$$f^l(x) > u \quad \text{for } l = 1, 2, \dots, m-1,$$

then

$$(\mu_k * f)^l(y) > \xi_{I, [a_k, b_k]}(u) \quad \text{for } l = 1, 2, \dots, 2m - 1.$$

PROPOSITION 3. Suppose that $f, f': I \rightarrow I$ are continuous maps with fixed points 0 and 1.

(1) If there is some $M > 0$ such that

$$|f(x) - f'(x)| < M \quad \text{for all } x \in I,$$

then

$$|\mu_k * f(y) - \mu_k * f'(y)| < M/2^k \quad \text{for all } y \in I.$$

(2) If f and f' coincide on $I - (u, v)$, where $[u, v]$ is a subinterval of I , then $\mu_k * f$ and $\mu_k * f'$ coincide on

$$I - (\xi_{I, [a_k, b_k]}(u), \xi_{I, [a_k, b_k]}(v)).$$

PROOF. (1) If $y \in I - (a_k, b_k)$, then $\mu_k * f(y) - \mu_k * f'(y) = 0$, and if $y \in [a_k, b_k]$, then

$$\begin{aligned} \mu_k * f(y) - \mu_k * f'(y) &= \xi_{I, [c_k, d_k]}(f(\xi_{[a_k, b_k], I}(y))) \\ &\quad - \xi_{I, [c_k, d_k]}(f'(\xi_{[a_k, b_k], I}(y))) \\ &= \xi_{I, [c_k, d_k]}(f(x)) - \xi_{I, [c_k, d_k]}(f'(x)) \\ &= (d_k - c_k)(f(x) - f'(x)) \\ &= \frac{1}{2^k}(f(x) - f'(x)), \end{aligned}$$

where $x = \xi_{[a_k, b_k], I}(y)$. Hence for any $y \in I$,

$$|\mu_k * f(y) - \mu_k * f'(y)| < M/2^k.$$

(2) By the definition, $\mu_k * f$ and $\mu_k * f'$ coincide on $I - (a_k, b_k)$. Since f and f' coincide on $I - (u, v)$, $\mu_k * f$ and $\mu_k * f'$ coincide on

$$\begin{aligned} \xi_{[a_k, b_k], I}^{-1}(I - (u, v)) &= \xi_{I, [a_k, b_k]}(I) - \xi_{I, [a_k, b_k]}((u, v)) \\ &= [a_k, b_k] - (\xi_{I, [a_k, b_k]}(u), \xi_{I, [a_k, b_k]}(v)). \end{aligned}$$

Therefore, $\mu_k * f$ and $\mu_k * f'$ coincide on

$$I - (\xi_{I, [a_k, b_k]}(u), \xi_{I, [a_k, b_k]}(v)).$$

3. Let

$$\begin{aligned} \alpha_3 &= \frac{1}{2^3}, \\ \alpha_4 &= \frac{1}{2^3} + \frac{1}{2} \cdot \frac{1}{2^4}, \\ \alpha_5 &= \frac{1}{2^3} + \frac{1}{2} \cdot \frac{1}{2^4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{2^5}, \\ &\dots\dots\dots \\ \alpha_k &= \frac{1}{2^3} + \frac{1}{2} \cdot \frac{1}{2^4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{2^5} + \dots + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{2^{k-3} - 1}{2^{k-3}} \cdot \frac{1}{2^k}, \\ &\dots\dots\dots \end{aligned}$$

and let

$$\begin{aligned}\beta_3 &= 1 - \frac{3}{2^3}, \\ \beta_4 &= 1 - \frac{3}{2^3} - \frac{1}{2} \cdot \frac{3}{2^4}, \\ \beta_5 &= 1 - \frac{3}{2^3} - \frac{1}{2} \cdot \frac{3}{2^4} - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{2^5}, \\ &\dots\dots\dots \\ \beta_k &= 1 - \frac{3}{2^3} - \frac{1}{2} \cdot \frac{3}{2^4} - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{2^5} - \dots - \frac{1}{2} \cdot \frac{3}{4} \dots \frac{2^{k-3} - 1}{2^{k-3}} \cdot \frac{3}{2^k}, \\ &\dots\dots\dots\end{aligned}$$

Obviously, $\alpha_3 < \alpha_4 < \alpha_5 < \dots < \alpha_{k-1} < \alpha_k < \dots$ and $\beta_3 > \beta_4 > \beta_5 > \dots > \beta_{k-1} > \beta_k > \dots$.

Since both sequences $\alpha_3, \alpha_4, \dots$ and $\beta_3, \beta_4, \dots$ are monotone and bounded, their limits exist. Let $\alpha = \lim_{k \rightarrow \infty} \alpha_k$ and $\beta = \lim_{k \rightarrow \infty} \beta_k$. By a simple calculation, we have that

$$\beta_k - \alpha_k = \frac{1}{2} \cdot \frac{3}{4} \dots \frac{2^{k-2} - 1}{2^{k-2}}.$$

Thus,

$$\beta - \alpha = \lim_{k \rightarrow \infty} (\beta_k - \alpha_k) = \prod_{i=1}^{\infty} \left(1 - \frac{1}{2^i}\right) > 0$$

(see [4]). Hence,

(*)

$$\alpha_3 < \alpha_4 < \dots < \alpha_{k-1} < \alpha_k < \dots < \alpha < \beta < \dots < \beta_k < \beta_{k-1} < \dots < \beta_4 < \beta_3.$$

By the definition of the notation, $\xi_{I,[u,v]}, \xi_{I,[u,v]}(x) = u + (v - u)x$ for any $x \in I$. Therefore,

$$\begin{aligned}\xi_{I,[a_3,b_3]} \circ \xi_{I,[a_4,b_4]} \circ \dots \circ \xi_{I,[a_{k-1},b_{k-1}]}(a_k) \\ &= a_3 + (b_3 - a_3)\xi_{I,[a_4,b_4]} \circ \dots \circ \xi_{I,[a_{k-1},b_{k-1}]}(a_k) \\ &= \dots \\ &= a_3 + (b_3 - a_3)a_4 + \dots + (b_3 - a_3)(b_4 - a_4) \dots (b_{k-1} - a_{k-1})a_k \\ &= \frac{1}{2^3} + \frac{1}{2} \cdot \frac{1}{2^4} + \dots + \frac{1}{2} \cdot \frac{3}{4} \dots \frac{2^{k-3} - 1}{2^{k-3}} \cdot \frac{1}{2^k} \\ &= \alpha_k.\end{aligned}$$

In brief,

$$(**)_1 \quad \alpha_k = \xi_{I,[a_3,b_3]} \circ \xi_{I,[a_4,b_4]} \circ \dots \circ \xi_{I,[a_{k-1},b_{k-1}]}(a_k),$$

and by a similar calculation,

$$(**)_2 \quad \beta_k = \xi_{I,[a_3,b_3]} \circ \xi_{I,[a_4,b_4]} \circ \dots \circ \xi_{I,[a_{k-1},b_{k-1}]}(b_k).$$

4. Now, we define

$$\begin{aligned}\chi_3 &= \mu_3, \\ \chi_4 &= \mu_3 * \mu_4, \\ \chi_5 &= \mu_3 * (\mu_4 * \mu_5), \\ &\dots\dots\dots \\ \chi_k &= \mu_3 * (\mu_4 * (\dots * (\mu_{k-1} * \mu_k) \dots)), \\ &\dots\dots\dots\end{aligned}$$

PROPOSITION 4. (1) χ_k is of type 2^∞ .

(2) $\chi_{k+1}|_{I-(\alpha_k, \beta_k)} = \chi_k|_{I-(\alpha_k, \beta_k)}$.

Therefore, if $l \geq k$, then

$$\chi_k|_{I-(\alpha_k, \beta_k)} = \chi_l|_{I-(\alpha_k, \beta_k)}.$$

(3) α_k is a periodic point of χ_k with period 2^{k-2} , and

$$(\chi_k)^m(\alpha_k) > \beta_k \quad \text{for } m = 1, 2, \dots, 2^{k-2} - 1.$$

Therefore, if $l \geq k$, then α_k is a periodic point of χ_l with period 2^{k-2} , and

$$(\chi_l)^m(\alpha_k) > \beta_k \quad \text{for } m = 1, 2, \dots, 2^{k-2} - 1.$$

(4) For any $x \in I$,

$$|\chi_{k+1}(x) - \chi_k(x)| < 2^{-(k+3)(k-2)/2}.$$

PROOF. (1) Propositions 1 and 2(1) imply the statement (1) of this proposition.

(2) By the definition, $\mu_k * \mu_{k+1}$ and μ_k coincide on $I - (a_k, b_k)$. By Proposition 3(2), $\mu_{k-1} * (\mu_k * \mu_{k+1})$ and $\mu_{k-1} * \mu_k$ coincide on

$$I - (\xi_{I, [a_{k-1}, b_{k-1}]}(a_k), \xi_{I, [a_{k-1}, b_{k-1}]}(b_k)).$$

Applying Proposition 3(2) repeatedly, we have that χ_{k+1} and χ_k coincide on

$$I - (\xi_{I, [a_3, b_3]} \circ \xi_{I, [a_4, b_4]} \circ \dots \circ \xi_{I, [a_{k-1}, b_{k-1}]}(a_k), \xi_{I, [a_3, b_3]} \circ \xi_{I, [a_4, b_4]} \circ \dots \circ \xi_{I, [a_{k-1}, b_{k-1}]}(b_k)),$$

which is $I - (\alpha_k, \beta_k)$ by $(**)1$ and $(**)2$.

(3) a_k is a periodic point of μ_k with period 2 and $\mu_k(a_k) = c_k > b_k$. By Proposition 2(2), $\xi_{I, [a_{k-1}, b_{k-1}]}(a_k)$ is a periodic point of $\mu_{k-1} * \mu_k$ with period 4, and

$$(\mu_{k-1} * \mu_k)^m(\xi_{I, [a_{k-1}, b_{k-1}]}(a_k)) > \xi_{I, [a_{k-1}, b_{k-1}]}(b_k)$$

for $m = 1, 2, 3$.

Applying Proposition 2(2) repeatedly, we have $\alpha_k = \xi_{I, [a_3, b_3]} \circ \xi_{I, [a_4, b_4]} \circ \dots \circ \xi_{I, [a_{k-1}, b_{k-1}]}(a_k)$ is a periodic point of χ_k with a period of 2^{k-2} , and

$$(\chi_k)^m(\alpha_k) > \xi_{I, [a_3, b_3]} \circ \xi_{I, [a_4, b_4]} \circ \dots \circ \xi_{I, [a_{k-1}, b_{k-1}]}(b_k) = \beta_k$$

for $m = 1, 2, \dots, 2^{k-2} - 1$.

(4) By the definition, for any $x \in I$

$$|\mu_k * \mu_{k+1}(x) - \mu_k(x)| < 2^{-k}.$$

By Proposition 3(1), for any $x \in I$

$$|\mu_{k-1} * (\mu_k * \mu_{k+1})(x) - \mu_{k-1} * \mu_k(x)| < 2^{-(k-1)} \cdot 2^{-k}.$$

Applying Proposition 3(1) repeatedly, we have that for any $x \in I$

$$\begin{aligned} |\chi_{k+1}(x) - \chi_k(x)| &= |\mu_3 * (\mu_4 * (\cdots * (\mu_k * \mu_{k+1}) \cdots))(x) \\ &\quad - \mu_3 * (\mu_k * (\cdots * (\mu_{k-1} * \mu_k) \cdots))(x)| \\ &< 2^{-3} \cdot 2^{-4} \cdots 2^{-k} \\ &= 2^{-(k+3)(k-2)/2}. \end{aligned}$$

5. By Proposition 4(4) the sequence of the maps $\chi_3, \chi_4, \dots$ is uniformly convergent, and its limit map, denoted by χ_∞ , is continuous. By a theorem of Block (see Main Theorem in [3]), χ_∞ is of type 2^∞ because all χ_k 's are of type 2^∞ .

Now, we will state and prove our main conclusion as follows.

THEOREM. *The set of recurrent points of the continuous map $\chi_\infty: I \rightarrow I$, which is of type 2^∞ , is not closed.*

PROOF. By Proposition 4(2) and the definition of χ_∞ it is easy to see that

$$\chi_\infty|_{I-(\alpha_k, \beta_k)} = \chi_k|_{I-(\alpha_k, \beta_k)}.$$

Hence, by Proposition 4(3), α_k is periodic of χ_∞ with period 2^{k-2} , and

$$\chi_\infty^m(\alpha_k) > \beta_k > \beta \quad \text{for } m = 1, 2, \dots, 2^{k-2} - 1.$$

Consider the accumulation point $\alpha = \lim_{k \rightarrow \infty} \alpha_k$ of periodic points of χ_∞ . For any positive integer $m > 0$

$$\chi_\infty^m(\alpha) = \lim_{k \rightarrow \infty} \chi_\infty^m(\alpha_k).$$

Thus, $\chi_\infty^m > \beta$ because $\chi_\infty^m(\alpha_{m+2}) > \beta$, $\chi_\infty^m(\alpha_{m+3}) > \beta, \dots$. Since $\alpha < \beta$ (see (*)), α is not a recurrent point of χ_∞ . This shows that the set of recurrent points is not closed.

REFERENCES

1. Xiong Jincheng, *The periods of periodic points of continuous self-maps of the interval whose recurrent points form a closed set*, J. China Univ. Sci. Tech. **13** (1983), 134–135.
2. A. M. Bloh, *The asymptotic behaviour of one-dimensional dynamical systems*, Uspekhi Mat. Nauk **37** (1982), 175–176.
3. L. Block, *Stability of periodic orbits in the theorem of Šarkovski*, Proc. Amer. Math. Soc. **81** (1981), 335–336.
4. A. Weil, *Two lectures on number theory, past and present*, Enseign. Math. **20** (1974), 87–110.

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