A PROPERTY OF WEAKLY COMPACT OPERATORS ON $C[0, 1]$

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ABSTRACT. It is shown that if T is a bounded linear operator on $C[0, 1]$ then either $\|I + T\|$ or $\|I - T\|$ equals $1 + \|T\|$. If T is a weakly compact operator then $\|I + T\| = 1 + \|T\|$, an extension of a result of Daugavet concerning compact operators on $C[0, 1]$.

In [2] Daugavet proved the following theorem: *If T is a compact operator on $C[0, 1]$ then $\|I + T\| = 1 + \|T\|$.* Later, Babenko and Pichugov [1] showed that the same is true for a compact operator on $L^1[0, 1]$. In general, if X is a Banach space and $T \in \mathcal{L}(X)$ then T is said to satisfy *Daugavet's equation* if $\|I - T\| = 1 + \|T\|$ [4]. This property has been found useful in work on a variety of problems in approximation theory (e.g. [4], and the references cited in [1]) and is of interest in the study of the metric geometry of Banach spaces of linear operators.

The purpose of this note is to observe that any weakly compact operator on $C[0, 1]$ also satisfies Daugavet's equation. More generally we prove the following result:

THEOREM. *Let T be a bounded linear operator on $C[0, 1]$. Then*

- (i) *either T or $(-T)$ satisfies Daugavet's equation;*
- (ii) *if T is weakly compact, T satisfies Daugavet's equation; i.e. $\|I + T\| = 1 + \|T\|$.*

Before proving this theorem we first note a simple result concerning normalized functions of bounded variation.

LEMMA. *Let $\alpha(t)$ be in $\text{NBV}[0, 1]$, s_0 in $(0, 1)$, and*

$$\mu_{s_0}(t) = \begin{cases} 0 & \text{if } 0 \leq t < s_0 \\ 1 & \text{if } s_0 \leq t \leq 1 \end{cases}.$$

If $\alpha(s_0-) \leq \alpha(s_0)$ then $\text{Var}(\mu_{s_0} + \alpha) = 1 + \text{Var}(\alpha)$.

PROOF. Clearly, $\text{Var}(\mu_{s_0} + \alpha) \leq 1 + \text{Var}(\alpha)$. Let $\varepsilon > 0$ be given and choose a partition $(t_i)_{i=0}^n$ of $[0, 1]$ for which

$$\sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})| > \text{Var}(\alpha) - \frac{\varepsilon}{2}$$

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and for which $s_0 \in (t_{k-1}, t_k)$ for some k . Since $\alpha(s_0-) \leq \alpha(s_0)$ there is a point p for which $t_{k-1} < p < s_0$ and $\alpha(s_0) - \alpha(p) > -\varepsilon/4$. It follows that

$$\begin{aligned} \text{Var}(\mu_{s_0} + \alpha) &\geq \sum_{i \neq k} |\alpha(t_i) - \alpha(t_{i-1})| \\ &\quad + |\alpha(p) - \alpha(t_{k-1})| + |1 + \alpha(s_0) - \alpha(p)| + |\alpha(t_k) - \alpha(s_0)| \\ &= \sum_{i \neq k} |\alpha(t_i) - \alpha(t_{i-1})| + |\alpha(p) - \alpha(t_{k-1})| + |\alpha(s_0) - \alpha(p)| \\ &\quad + |\alpha(t_k) - \alpha(s_0)| - |\alpha(s_0) - \alpha(p)| + |1 + \alpha(s_0) - \alpha(p)| \\ &\geq \sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})| - \frac{\varepsilon}{4} + 1 - \frac{\varepsilon}{4} \geq 1 + \text{Var}(\alpha) - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $\text{Var}(\mu_{s_0} + \alpha) \geq 1 + \text{Var}(\alpha)$, and the lemma is proved.

PROOF OF THE THEOREM. As in [3, p. 517] we can represent T as the integral operator $(Tf)(s) = \int_0^1 f(t) dk(s, t)$, where $K(s, \cdot)$ is in NBV[0, 1] for all $s \in [0, 1]$ and

$$\|T\| = \sup_s \|K(s, \cdot)\|_{\text{NBV}} = \sup_s \text{Var } K(s, \cdot).$$

In fact, identifying $C[0, 1]^*$ and NBV[0, 1] (as usual), for every s in $(0, 1)$, $K(s, \cdot) = T^*(\mu_s)$, where $\mu_s(t)$ is the step function defined in the preceding lemma. From this we see that $\|K(s, \cdot)\|_{\text{NBV}}$ is a lower semicontinuous function on $[0, 1]$.

It is now a simple matter to see that (i) holds. Let $\varepsilon > 0$ be given and $s_0 \in (0, 1)$ a point for which $\|K(s_0, \cdot)\|_{\text{NBV}} > \|T\| - \varepsilon/2$. Suppose $K(s_0, s_0-) \leq K(s_0, s_0)$. Then by the Lemma (with $\alpha(t) = K(s_0, t)$) we have $\|\mu_{s_0} + K(s_0, \cdot)\|_{\text{NBV}} = 1 + \|K(s_0, \cdot)\|_{\text{NBV}} > 1 + \|T\| - \varepsilon/2$. Since NBV[0, 1] = $C[0, 1]^*$ there is an f in $C[0, 1]$ with $\|f\| = 1$ for which

$$\langle \mu_{s_0}, f \rangle + \int_0^1 f(t) dK(s_0, t) > \|\mu_{s_0} + K(s_0, \cdot)\|_{\text{NBV}} - \frac{\varepsilon}{2} > 1 + \|T\| - \varepsilon.$$

But then

$$\|f + Tf\| = \sup_s \left[f(s) + \int_0^1 f(t) dK(s, t) \right] > f(s_0) + \int_0^1 f(t) dK(s_0, t) > 1 + \|T\| - \varepsilon,$$

so $\|I + T\| > 1 + \|T\| - \varepsilon$. On the other hand, if $K(s_0, s_0-) > K(s_0, s_0)$, then $-K(s_0, s_0-) < -K(s_0, s_0)$, so by the same argument (applied to $\alpha(t) = -K(s_0, t)$) we get $\|I - T\| > 1 + \|T\| - \varepsilon$. This is true for each $\varepsilon > 0$, so let $\varepsilon_n = 1/n$, $n = 1, 2, \dots$, and for each such n choose s_n in $(0, 1)$ for which $\|K(s_n, \cdot)\|_{\text{NBV}} > \|T\| - \varepsilon_n/2$. As we showed above, for each n either $\|I + T\| > 1 + \|T\| - 1/2n$ or $\|I - T\| > 1 + \|T\| - 1/2n$. By considering subsequences it follows that either $\|I + T\| \geq 1 + \|T\|$ or $\|I - T\| \geq 1 + \|T\|$, and (i) is proved.

Now suppose T is a weakly compact operator. Again, given any $\varepsilon > 0$ choose s_0 in $(0, 1)$ for which $\|K(s_0, \cdot)\|_{\text{NBV}} > \|T\| - \varepsilon/4$. Since $\|K(s, \cdot)\|_{\text{NBV}}$ is lower semicontinuous on $[0, 1]$ there exists a $\delta > 0$ such that if $|s - s_0| < \delta$ then $\|K(s, \cdot)\|_{\text{NBV}} \geq \|K(s_0, \cdot)\|_{\text{NBV}} - \varepsilon/4 > \|T\| - \varepsilon/2$. Moreover, since T^* is weakly compact on NBV[0, 1] [3, p. 485] and $K(s, \cdot) = T^*(\mu_s)$ for s in $(0, 1)$ we see that $\{K(s, \cdot)\}_s$ in $(0, 1)$ is a weakly compact subset of NBV[0, 1], and hence a weakly sequentially compact set [3, p. 430]. It follows that there is a function $g(t)$ in

NBV[0, 1] with the property that for every $\delta > 0$ there exists a $\beta > 0$ so that, if r and w are in $(0, 1)$ and $|g(r) - g(w)| < \beta$, then $|K(s, r) - K(s, w)| < \delta$ for every s in $(0, 1)$ [3, p. 343]. In particular, if $g(t)$ is continuous at a point $t = s_1$, $k(s, t)$ is also continuous at $t = s_1$ for every s in $(0, 1)$.

Since $g(t)$ is continuous at all but (possibly) a countable number of points in $(0, 1)$ we can find (for the $\delta > 0$ specified above) a point $s_1 \in (0, 1)$ for which $|s_0 - s_1| < \delta$ and at which $g(t)$ is continuous. But then, as we showed above, $K(s_1, t)$ is also continuous at $t = s_1$, so $K(s_1, s_1-) = K(s_1, s_1)$ and by the Lemma and the choice of s_1 , $\|\mu_{s_1} + K(s_1, \cdot)\|_{\text{NBV}} = 1 + \|K(s_1, \cdot)\|_{\text{NBV}} \geq 1 + \|T\| - \varepsilon/2$. As in the proof of (i) we now take f in $C[0, 1]$ with $\|f\| = 1$ for which $\langle \mu_{s_1} + K(s_1, \cdot), f \rangle > \|\mu_{s_1} + K(s_1, \cdot)\| - \varepsilon/2$, and hence for which $f(s_1) + \int_0^1 f(t) dK(s_1, t) > 1 + \|T\| - \varepsilon$. But then $\|I + T\| = \sup_{\|h\|=1} \|h + Th\| \geq f(s_1) + \int_0^1 f(t) dK(s_1, t) > 1 + \|T\| - \varepsilon$. Since $\varepsilon > 0$ is arbitrary we conclude that $\|I + T\| = 1 + \|T\|$, and (ii) is also proved.

REMARK. It is natural to ask whether the results of this theorem can be extended to the space $C(S)$ where S is an arbitrary compact H space. By an argument entirely analogous to that of the Lemma above (using the identification of $C(S)^*$ and $\text{rca}(S)$ instead of the identification of $C[0, 1]^*$ and NBV[0, 1] used in the Lemma [3, p. 265]) one can again show that if T is any bounded linear operator on $C(S)$ then either $\|I + T\|$ or $\|I - T\|$ equals $1 + \|T\|$. However, if S has at least one isolated point s_0 then $S = K \cup \{s_0\}$, where K is compact, and we can identify $C(S)$ with $X = \{(f, a) | f \in C(K), a \in R\}$ and $\|(f, a)\| = \max\{\|f\|, |a|\}$. Thus if $T: X \rightarrow X$ is defined by $T(f, a) = (0, -a)$ then $\|T\| = 1$, T is compact (in fact, 1-dimensional), but $\|I + T\| = 1 \neq 1 + \|T\|$. It follows that there is a 1-dimensional operator on $C(S)$ with the same property.

As a corollary to these observations and the fact that if μ is a positive measure $L^\infty(\mu)$ is isometrically isomorphic to $C(S)$ for S an extremally disconnected compact Hausdorff space [3, pp. 398, 445], we see that if T is a bounded linear operator on $L^\infty(\mu)$ then $\|I + T\|$ or $\|I - T\|$ equals $1 + \|T\|$, but there is always a 1-dimensional operator T on $L^\infty(\mu)$ for which $\|T\| = 1$ but $\|I + T\| \neq 1 + \|T\|$.

ADDED IN PROOF. After this paper was accepted the author learned of the paper *A property of compact operators*, Proc. Amer. Math. Soc. **91** (1984), 231-236, by Herbert Kamowitz, which also deals with Daugavet's equation.

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