ON A THEOREM OF HARDY AND LITTLEWOOD ON THE POLYDISC¹

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Dedicated to Professor Jae Kyu Lim on his sixtieth birthday

ABSTRACT. We prove the polydisc version of the theorem of Hardy and Littlewood on the fractional integral: If $0 < \alpha < 1/p$ and if $f \in H^p$, then $I^\alpha f \in H^q$ with $q = p/(1 - \alpha p)$ where $I^\alpha f$ is the fractional integral of f of order α .

0. Introduction. If U denotes the open unit disc in the complex plane $\mathbb{C}$, the Hardy space $H^p(U)$, $0 < p < \infty$, consists of all functions f holomorphic in U for which

$$\|f\|_p = \sup_{0 \leq r < 1} M_p(r; f) < \infty,$$

where

$$M_p(r; f) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

See [1] for the theory of Hardy spaces. If $f(z) = \sum a_k z^k$ is holomorphic in U , the fractional integral of f of order $\alpha > 0$ is defined as

$$I^\alpha f(z) = \sum \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} a_k z^k = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\rho)^{\alpha-1} f(\rho z) d\rho.$$

This definition differs from that of Hardy and Littlewood in [4] by a factor z^α . Flett [2] used $\log(1/\rho)$ in place of $(1-\rho)$ above for his definition. Since $\log(1/\rho) \sim 1-\rho$ as $\rho \rightarrow 1$, we see that all these different definitions are "equivalent" for our purpose.

The following theorem was proved originally by Hardy and Littlewood in [4, 5] and by Flett in [2] with his definition of fractional integral.

THEOREM. *If $0 < \alpha < 1/p$ and if $f \in H^p$, then $I^\alpha f \in H^q$ with $q = p/(1 - \alpha p)$.*

A new proof was given to an extended version of this theorem in [7]. The purpose of the present paper is to prove the analogue of this theorem for the polydisc U^n of $\mathbb{C}^n$. The idea is to iterate the one variable proof of [7]. We only consider the case $n = 2$, but the procedure can be repeated for arbitrary n .

A holomorphic function f in U^2 belongs to the Hardy space $H^p(U^2)$ if and only if

$$\|f\|_p = \sup_{0 \leq r, s < 1} M_p(r, s; f) < \infty,$$

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where

$$M_p(r, s; f) = \left(\int_0^{2\pi} \int_0^{2\pi} |f(re^{i\theta}, se^{i\eta})|^p \frac{d\theta}{2\pi} \frac{d\eta}{2\pi} \right)^{1/p}.$$

See [8] for the general theory of Hardy spaces $H^p(U^n)$. The fractional integral of f of order $\alpha > 0$ is defined as

$$I^{\alpha, \alpha} f(z, w) = \frac{1}{\Gamma(\alpha)^2} \int_0^1 \int_0^1 (1-\rho)^{\alpha-1} (1-\sigma)^{\alpha-1} f(\rho z, \sigma w) d\rho d\sigma.$$

$f(\cdot, w)$ denotes the function $z \mapsto f(z, w)$ with w fixed and $f_{r,s}$ the function $(z, w) \mapsto f(rz, sw)$. Throughout this paper, $C(\cdots)$ denotes a positive constant depending only on the argument $(\cdots)$ and it may vary from occurrence to occurrence even in the proof of the same theorem.

1. Two more theorems of Hardy and Littlewood on U^2 . For the proof of our main theorem for the case $q \leq 1$, we need two other theorems of Hardy and Littlewood on the polydisc.

1.1. LEMMA. If $f \in H^p(U^2)$, then

$$|f(z, w)| \leq C(p) \|f(z, \cdot)\|_p (1 - |w|)^{-1/p},$$

and

$$|f(z, w)| \leq C(p) \|f(\cdot, w)\|_p (1 - |z|)^{-1/p}.$$

PROOF. This is a well-known theorem of Hardy and Littlewood [1, Theorem 5.9; 4, Theorem 27] applied to the one variable function $f(z, \cdot)$ with z fixed and $f(\cdot, w)$ with w fixed.

1.2. LEMMA. If $f \in H^p(U^2)$, then

$$\|f(\cdot, w)\|_p \leq C(p) \|f\|_p (1 - |w|)^{-1/p}.$$

PROOF.

$$\begin{aligned} \|f(\cdot, w)\|_p^p &= \lim_{r \rightarrow 1} \int_0^{2\pi} |f(re^{i\theta}, w)|^p \frac{d\theta}{2\pi} \\ &\leq C(p)(1 - |w|)^{-1} \lim_{r \rightarrow 1} \int_0^{2\pi} \|f(re^{i\theta}, \cdot)\|_p^p \frac{d\theta}{2\pi} \quad (\text{Lemma 1.1}) \\ &= C(p)(1 - |w|)^{-1} \lim_{r \rightarrow 1} \int_0^{2\pi} \lim_{s \rightarrow 1} \int_0^{2\pi} |f(re^{i\theta}, se^{i\eta})|^p \frac{d\eta}{2\pi} \frac{d\theta}{2\pi} \\ &= C(p)(1 - |w|)^{-1} \lim_{\substack{r \rightarrow 1 \\ s \rightarrow 1}} \int_0^{2\pi} \int_0^{2\pi} |f(re^{i\theta}, se^{i\eta})|^p \frac{d\eta}{2\pi} \frac{d\theta}{2\pi} \quad (\text{MCT}) \\ &= C(p)(1 - |w|)^{-1} \|f\|_p^p. \end{aligned}$$

Lemma 1.1 and monotone convergence theorem (MCT) are used above.

1.3. LEMMA. If $f \in H^p(U^2)$ and if $M(\eta) = \sup_{0 \leq s < 1} \|f(\cdot, se^{i\eta})\|_p$, then

$$\int_0^{2\pi} M(\eta)^p \frac{d\eta}{2\pi} \leq C(p) \|f\|_p^p.$$

PROOF.

$$\begin{aligned}
 \int_0^{2\pi} M(\eta)^p \frac{d\eta}{2\pi} &= \int_0^{2\pi} \sup_{0 \leq s < 1} \|f(\cdot, se^{i\eta})\|_p^p \frac{d\eta}{2\pi} \\
 &= \int_0^{2\pi} \sup_{0 \leq s < 1} \sup_{0 \leq r < 1} \int_0^\pi |f(re^{i\theta}, se^{i\eta})|^p \frac{d\theta}{2\pi} \frac{d\eta}{2\pi} \\
 &\leq \int_0^{2\pi} \int_0^{2\pi} \sup_{0 \leq r, s < 1} |f(re^{i\theta}, se^{i\eta})|^p \frac{d\theta}{2\pi} \frac{d\eta}{2\pi} \\
 &\leq C(p) \|f\|_p^p.
 \end{aligned}$$

We applied the complex maximal theorem on the polydisc proved by Zygmund [10] for the last inequality.

A more general version of the following theorem was proved by Frazier [3] and by Jawerth and Torchinsky [6]. For this special case we give still another proof by iterating Ahern's proof [7, Theorem B] of one variable version.

1.4. THEOREM. If $0 < p < q < \infty$ and if $f \in H^p(U^2)$, then

$$\int_0^1 \int_0^1 (1-r)^{q/p-2} (1-s)^{q/p-2} M_1(r, s; f)^q dr ds < \infty.$$

PROOF. For the moment, we fix w and set

$$(1.1) \quad M(\theta, w) = \sup_{0 \leq r < 1} |f(re^{i\theta}, w)|.$$

By Lemma 1.1 and (1.1), we have

$$(1.2) \quad \int_0^1 (1-r)^{q/p-2} |f(re^{i\theta}, w)|^q dr$$

$$(1.3) \quad \leq C(p, q) \|f(\cdot, w)\|_p^q \int_0^\lambda (1-r)^{-2} dr + M(\theta, w)^q \int_\lambda^1 (1-r)^{q/p-2} dr.$$

If $M(\theta, w) \leq \|f(\cdot, w)\|_p$, we set $\lambda = 0$ in (1.3). (1.2) is then dominated by $C(p, q) \|f(\cdot, w)\|_p^q$. If $M(\theta, w) > \|f(\cdot, w)\|_p$, we set $\lambda = 1 - (\|f(\cdot, w)\|_p / M(\theta, w))^p$ in (1.3). (1.2) is then dominated by $C(p, q) \|f(\cdot, w)\|_p^{q-p} M(\theta, w)^p$. Hence, we have, for any θ ,

$$\begin{aligned}
 (1.4) \quad &\int_0^1 (1-r)^{q/p-2} |f(re^{i\theta}, w)|^q dr \\
 &\leq C(p, q) (\|f(\cdot, w)\|_p^q + \|f(\cdot, w)\|_p^{q-p} M(\theta, w)^p).
 \end{aligned}$$

We integrate (1.4) with respect to $d\theta/2\pi$ and use the complex maximal theorem applied to the one variable function $f(\cdot, w)$ to get

$$(1.5) \quad \int_0^1 \int_0^{2\pi} (1-r)^{q/p-2} |f(re^{i\theta}, w)|^p \frac{d\theta}{2\pi} dr \leq C(p, q) \|f(\cdot, w)\|_p^q.$$

Next, we set

$$(1.6) \quad M(\eta) = \sup_{0 \leq s < 1} \|f(\cdot, se^{i\eta})\|_p.$$

As we have done above, we have by Lemma 1.2 and (1.6)

$$(1.7) \quad \int_0^1 (1-s)^{q/p-2} \|f(\cdot, se^{i\eta})\|_p^q ds$$

$$(1.8) \quad \leq C(p, q) \|f\|_p^q \int_0^\lambda (1-s)^{-2} ds + M(\eta)^q \int_\lambda^1 (1-s)^{q/p-2} ds.$$

We set $\lambda = 0$ if $M(\eta) \leq \|f\|_p$ and $\lambda = 1 - (\|f\|_p/M(\eta))^p$ if $M(\eta) > \|f\|_p$ in (1.8). We have then, for any η ,

$$(1.9) \quad \int_0^1 (1-s)^{q/p-2} \|f(\cdot, se^{i\eta})\|_p^q ds \leq C(p, q) (\|f\|_p^q + \|f\|_p^{q-p} M(\eta)^p).$$

We integrate (1.9) with respect to $d\eta/2\pi$ and use Lemma 1.3 to get

$$(1.10) \quad \int_0^1 \int_0^{2\pi} (1-s)^{q/p-2} \|f(\cdot, se^{i\eta})\|_p^q \frac{d\eta}{2\pi} ds \leq C(p, q) \|f\|_p^q.$$

If we combine (1.5) and (1.10), we finish the proof.

The one variable version of the next theorem was proved by Hardy and Littlewood in [4].

1.5. THEOREM. If $0 < q \leq 1$ and if f is holomorphic in U^2 , then

$$\|I^{\alpha, \alpha} f\|_q^p \leq C(\alpha, q) \int_0^1 \int_0^1 (1-\rho)^{q\alpha-1} (1-\sigma)^{q\alpha-1} M_q^q(\rho, \sigma; f) d\rho d\sigma.$$

PROOF. If we define

$$I^{\alpha, 0} f(z, w) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\rho)^{\alpha-1} f(\rho z, w) d\rho,$$

we have

$$(1.11) \quad I^{\alpha, \alpha} f(z, w) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1} I^{\alpha, 0} f(z, \sigma w) d\sigma.$$

By applying the one variable version of this theorem to $I^{\alpha, 0} f$ with w fixed, we have

$$(1.12) \quad \int_0^{2\pi} |I^{\alpha, 0} f(re^{i\theta}, w)|^q \frac{d\theta}{2\pi} \leq C(\alpha, q) \int_0^1 (1-\rho)^{q\alpha-1} \int_0^{2\pi} |f(\rho re^{i\theta}, w)|^q \frac{d\theta}{2\pi} d\rho.$$

We can also apply the one variable version of this theorem to $I^{\alpha, \alpha} f$ with z fixed because of (1.11) and we get

$$(1.13) \quad \int_0^{2\pi} |I^{\alpha, \alpha} f(z, se^{i\eta})|^q \frac{d\eta}{2\pi} \leq C(\alpha, q) \int_0^1 (1-\sigma)^{q\alpha-1} \int_0^{2\pi} |I^{\alpha, 0} f(z, \sigma se^{i\eta})|^q \frac{d\eta}{2\pi} d\sigma.$$

If we combine (1.12) and (1.13) and use Fubini's theorem, we have

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2\pi} |I^{\alpha, \alpha} f(re^{i\theta}, se^{i\eta})|^q \frac{d\theta}{2\pi} \frac{d\eta}{2\pi} \\ & \leq C(\alpha, q) \int_0^1 \int_0^1 (1-\rho)^{q\alpha-1} (1-\sigma)^{q\alpha-1} M_q^q(\rho r, \sigma s; f) d\rho d\sigma \\ & \leq C(\alpha, q) \int_0^1 \int_0^1 (1-\rho)^{q\alpha-1} (1-\sigma)^{q\alpha-1} M_q^q(\rho, \sigma; f) d\rho d\sigma \end{aligned}$$

for any $0 \leq r, s < 1$. This completes the proof.

2. Main theorem.

2.1. LEMMA. If $f \in H^p(U^2)$ and if

$$M(\eta) = \sup_{0 \leq \sigma < 1} \left(\int_0^{2\pi} |f(re^{i\theta}, \sigma se^{i\eta})|^p \frac{d\theta}{2\pi} \right)^{1/p},$$

then $\int_0^{2\pi} M(\eta)^p d\eta/2\pi \leq C(p)M_p(r, s; f)^p$.

PROOF.

$$\begin{aligned} \int_0^{2\pi} M(\eta)^p \frac{d\eta}{2\pi} &= \int_0^{2\pi} \sup_{0 \leq \sigma < 1} \int_0^{2\pi} |f(re^{i\theta}, \sigma se^{i\eta})|^p \frac{d\theta}{2\pi} \frac{d\eta}{2\pi} \\ &\leq \int_0^{2\pi} \int_0^{2\pi} \sup_{0 \leq \sigma < 1} |f(re^{i\theta}, \sigma se^{i\eta})|^p \frac{d\theta}{2\pi} \frac{d\eta}{2\pi} \\ &= \int_0^{2\pi} \left(\int_0^{2\pi} \sup_{0 \leq \sigma < 1} |f(re^{i\theta}, \sigma se^{i\eta})|^p \frac{d\eta}{2\pi} \right) \frac{d\theta}{2\pi} \quad (\text{Fubini}) \\ &\leq C(p) \int_0^{2\pi} \int_0^{2\pi} |f(re^{i\theta}, se^{i\eta})|^p \frac{d\eta}{2\pi} \frac{d\theta}{2\pi} \\ &= C(p)M_p(r, s; f)^p. \end{aligned}$$

We applied the complex maximal theorem to the one variable function $f(z, \cdot)$ to get the last inequality.

Now we are ready to prove our main theorem:

2.2. THEOREM. If $0 < \alpha < 1/p$ and if $f \in H^p(U^2)$, then $I^{\alpha, \alpha} f \in H^q(U^2)$ with $q = p/(1 - \alpha p)$.

PROOF. The case $q \leq 1$ follows from Theorems 1.4 and 1.5 since $q/p - 2 = \alpha q - 1$. Now, we assume $q \geq 1$ and set

$$(2.1) \quad M(re^{i\theta}, \sigma w) = \sup_{0 \leq \rho < 1} |f(\rho re^{i\theta}, \sigma w)|.$$

We write $z = re^{i\theta}$ and $w = se^{i\eta}$. By Lemma 1.1,

$$(2.2) \quad |f(\rho z, \sigma w)| \leq C(p) \|f_{r,s}(\cdot, \sigma se^{i\eta})\|_p (1 - \rho)^{-1/p}.$$

By (2.1) and (2.2), we have, as in the proof of Theorem 1.4,

$$\begin{aligned} (2.3) \quad &\int_0^1 (1 - \rho)^{\alpha-1} |f(\rho re^{i\theta}, \sigma se^{i\eta})| d\rho \\ &\leq C(p) \|f_{r,s}(\cdot, \sigma se^{i\eta})\|_p \int_0^\lambda (1 - \rho)^{\alpha-1/p-1} d\rho \end{aligned}$$

$$(2.4) \quad + M(re^{i\theta}, \sigma w) \int_\lambda^1 (1 - \rho)^{\alpha-1} d\rho.$$

If $\|f_{r,s}(\cdot, \sigma se^{i\eta})\|_p \geq M(re^{i\theta}, \sigma w)$, we set $\lambda = 0$ in (2.4). (2.3) is then dominated by $C(\alpha) \|f_{r,s}(\cdot, \sigma se^{i\eta})\|_p$. If $\|f_{r,s}(\cdot, \sigma se^{i\eta})\|_p < M(re^{i\theta}, \sigma w)$, we set

$$\lambda = 1 - (\|f_{r,s}(\cdot, \sigma se^{i\eta})\|_p / M(re^{i\theta}, \sigma w))^p.$$

(2.3) is then dominated by

$$C(\alpha, p) \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p^{\alpha p} M(re^{i\theta}, \sigma w)^{1-\alpha p}.$$

Hence, for any η we have

$$(2.5) \quad \int_0^1 (1-\rho)^{\alpha-1} |f(\rho re^{i\theta}, \sigma se^{i\eta})| d\rho \\ \leq C(\alpha, p) (\|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p + \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p^{\alpha p} M(re^{i\theta}, \sigma w)^{1-\alpha p}).$$

Integrating (2.5) with respect to $(1-\sigma)^{\alpha-1} d\sigma$, we get

$$(2.6) \quad |I^{\alpha, \alpha} f(z, w)| \leq C(\alpha, p) \left(\int_0^1 (1-\sigma)^{\alpha-1} \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p d\sigma \right. \\ \left. + \int_0^1 M(re^{i\theta}, \sigma w)^{1-\alpha p} \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p^{\alpha p} (1-\sigma)^{\alpha-1} d\sigma \right).$$

We take the q -mean on both sides of (2.6) with respect to $d\theta/2\pi$ and use the continuous form of Minkowski's inequality to get

$$(2.7) \quad \left(\int_0^{2\pi} |I^{\alpha, \alpha}(re^{i\theta}, se^{i\eta})|^q \frac{d\theta}{2\pi} \right)^{1/q} \\ \leq C(\alpha, p) \left(\int_0^1 (1-\sigma)^{\alpha-1} \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p d\sigma \right. \\ \left. + \int_0^1 \left(\int_0^{2\pi} M(re^{i\theta}, \sigma w)^p \frac{d\theta}{2\pi} \right)^{(1-\alpha p)/p} \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p^{\alpha p} (1-\sigma)^{\alpha-1} d\sigma \right) \\ \leq C(\alpha, p) \int_0^1 (1-\sigma)^{\alpha-1} \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p d\sigma.$$

We use the complex maximal theorem

$$\int_0^1 M(re^{i\theta}, \sigma se^{i\eta})^p \frac{d\theta}{2\pi} \leq C(p) \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p^p$$

to get the last inequality in (2.7). Next, we set

$$(2.8) \quad M(\eta) = \sup_{0 \leq \sigma < 1} \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p.$$

By Lemma 1.2, we have

$$(2.9) \quad \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p \leq C(p) \|f_{r,s}\|_p (1-\sigma)^{-1/p}.$$

By (2.8) and (2.9), we have as before

$$\int_0^1 \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p (1-\sigma)^{\alpha-1} d\sigma \\ \leq C(p) \|f_{r,s}\|_p \int_0^\lambda (1-\sigma)^{\alpha-1/p-1} d\sigma + M(\eta) \int_\lambda^1 (1-\sigma)^{\alpha-1} d\sigma.$$

We set $\lambda = 0$ if $M(\eta) \leq \|f_{r,s}\|_p$ and $\lambda = 1 - (\|f_{r,s}\|_p/M(\eta))^p$ otherwise. We then have for any η

$$\begin{aligned} \int_0^1 \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p (1-\sigma)^{\alpha-1} d\sigma \\ \leq C(\alpha, p) (\|f_{r,s}\|_p + \|f_{r,s}\|_p^{\alpha p} M(\eta)^{1-\alpha p}). \end{aligned}$$

If we take the q -mean on both sides with respect to $d\eta/2\pi$, we have

$$\begin{aligned} (2.10) \quad & \left(\int_0^{2\pi} \left(\int_0^1 \|f_{r,s}(\cdot, \sigma e^{i\eta})\|_p (1-\sigma)^{\alpha-1} d\sigma \right)^q \frac{d\eta}{2\pi} \right)^{1/q} \\ & \leq C(\alpha, p) \left(\|f_{r,s}\|_p + \|f_{r,s}\|_p^{\alpha p} \left(\int_0^{2\pi} M(\eta)^p d\eta \right)^{1/q} \right) \\ & \leq C(\alpha, p) \|f_{r,s}\|_p \end{aligned}$$

by Lemma 2.1. Combining (2.7) and (2.10) we have

$$M_q(r, s; I^{\alpha, \alpha} f) \leq C(\alpha, p) \|f_{r,s}\|_p;$$

so $\|I^{\alpha, \alpha} f\|_q \leq C(\alpha, p) \|f\|_p$. This completes the proof.

3. Remark. If f is holomorphic in the unit ball B_n of $\mathbb{C}^n$, we can define the radial fractional integral of f of order α as

$$I_r^\alpha f(z) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\rho)^{\alpha-1} f(\rho z) d\rho,$$

as in the one variable case. We refer to [8] for the theory of Hardy spaces $H^p(B_n)$ of the unit ball B_n . Since we have the estimate $|f(z)| \leq 2^{n/p} \|f\|_p (1-|z|)^{-n/p}$ if $f \in H^p(B_n)$ [8, Theorem 7.2.5], and the complex maximal theorem [8, Theorem 5.6.5] for the ball, we can prove as in the one variable case

3.1. THEOREM. If $0 < \alpha < n/p$ and if $f \in H^p(B_n)$, then $I_r^\alpha f \in H^q$ with $q = np/(n - \alpha p)$.

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