DENSE G_δ 'S CONTAIN ORTHONORMAL BASES

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ABSTRACT. We show that a dense G_δ in the unit sphere of a Hilbert space necessarily contains an orthonormal basis. This allows us to choose an orthonormal basis of cyclic and separating vectors for a standard von Neumann algebra.

In choosing an orthonormal basis for a Hilbert space, there are many occasions when one wants to choose the members of this basis to have some special property. Quite often the vectors with this property form a subspace, and then it is known that there is no problem as long as the subspace is dense (Proposition 1). However, there are occasions when the vectors of interest do not form a subspace, and the temptation is to give up, as a little reflection shows that a dense subset need not contain an orthonormal basis (Proposition 2). In this note we show that a dense G_δ set in the unit sphere of a Hilbert space must contain an orthonormal basis (Theorem 10). $\mathcal{H}$ denotes a separable infinite-dimensional complex Hilbert space, and $\mathcal{H}_1$ its unit sphere, the set of unit vectors.

PROPOSITION 1. *Any dense subspace of $\mathcal{H}$ contains an orthonormal basis.*

PROOF. Let $\mathcal{D}$ be a dense subspace. Then $\mathcal{D}_1$ (the unit sphere of $\mathcal{D}$) is dense in $\mathcal{H}_1$ and, being a subset of a separable metric space, $\mathcal{D}_1$ is separable. Choose a countable dense set E in $\mathcal{D}_1$; E is then also dense in $\mathcal{H}_1$, so the linear span of E is dense in $\mathcal{H}$. Applying the Gram-Schmidt procedure to E results in an orthonormal set in $\mathcal{D}$ whose linear span is still dense in $\mathcal{H}$, i.e. an orthonormal basis. $\square$

PROPOSITION 2. *There is a dense subset of $\mathcal{H}_1$ containing no orthonormal basis.*

PROOF. Let $\{U_n\}$ be a countable base for the topology of $\mathcal{H}_1$. Assume that $\xi_i \in U_i$, $i = 1, \dots, n-1$, have been chosen. The subspaces $[\xi_i]^\perp$, $i = 1, \dots, n-1$, are all nowhere dense in $\mathcal{H}$, and hence their intersections with $\mathcal{H}_1$ are nowhere dense. Choose $\xi_n \in U_n$ so that it does not lie in $[\xi_i]^\perp$, $i = 1, \dots, n-1$. The set $\{\xi_n\}$ is then dense in $\mathcal{H}_1$, but as ξ_m is never orthogonal to ξ_n , there can be no orthonormal basis in this set. $\square$

The following lemma has a simple set-theoretic proof.

LEMMA 3. *Let $f: X \rightarrow Y$ be a function from a set X to a set Y . Then if $U \subset X$ and $V \subset Y$, it is always true that $f(U \cap f^{-1}(V)) = f(U) \cap V$.*

LEMMA 4. *Let $f: X \rightarrow Y$ be a continuous open map. Then if S is a dense G_δ in Y , $f^{-1}(S)$ is a dense G_δ in X .*

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PROOF. Let V be an open dense set in Y . If U is an open set in X such that $U \cap f^{-1}(V)$ is empty, then by Lemma 3, $f(U \cap f^{-1}(V)) = f(U) \cap V$ is empty. Since $f(U)$ is open in Y , $f(U)$ must be empty by the density of V . Therefore U must be empty, showing that $f^{-1}(V)$ is an open dense set in X . Now suppose S is a dense G_δ in Y with $S = \bigcap_1^\infty V_n$, where each V_n is an open dense set in Y . Then $f^{-1}(S) = \bigcap_1^\infty f^{-1}(V_n)$, which is then a dense G_δ in X . $\square$

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{H}_1$ its unit sphere, and let $\mathcal{H}_1^\infty = \times_1^\infty \mathcal{H}_1^n$, each $\mathcal{H}_1^n$ a copy of $\mathcal{H}_1$. Then $\mathcal{H}_1^\infty$ is a complete metric space with the metric $\rho(\xi, \eta) = \sum_{n=1}^\infty 2^{-n} \|\xi_n - \eta_n\|$, where $\xi = \{\xi_n\}$ and $\eta = \{\eta_n\}$ [5, 24.11]. If $\{\xi_k\}$ is a sequence in $\mathcal{H}_1^\infty$ that converges to η , then $(\xi_k)_n$ converges in norm to η_n in $\mathcal{H}_1$ as $k \rightarrow \infty$ for each n . Let π_n denote the canonical projection map from $\mathcal{H}_1^\infty$ to $\mathcal{H}_1^n$, given by $\pi_n(\xi) = \xi_n$.

Define $\mathbf{S} = \mathbf{S}(\mathcal{H}) = \{\xi \in \mathcal{H}_1^\infty \mid \xi = \{\xi_n\} \text{ is an orthonormal sequence in } \mathcal{H}\}$. Then $\mathbf{S}$ is a closed subset of $\mathcal{H}_1^\infty$ due to the norm convergence of the individual entries mentioned above, and hence complete with respect to ρ .

LEMMA 5. $\pi_n|_{\mathbf{S}}$ is a continuous open map of $\mathbf{S}$ onto $\mathcal{H}_1$.

PROOF. Only the openness of the restriction is nontrivial. If $U \subset \mathbf{S}$ is open and $\xi \in U$, then $U \supset B_\rho(\xi, \varepsilon) \cap \mathbf{S}$ for some $\varepsilon > 0$, where $B_\rho(\xi, \varepsilon)$ represents the ball of radius ε about ξ in $\mathcal{H}_1^\infty$. Let $\zeta \in \mathcal{H}_1$ such that $\|\xi_n - \zeta\| < \varepsilon$. Then there exists η in $B_\rho(\xi, \varepsilon) \cap \mathbf{S}$ such that $\eta_n = \zeta$. In fact, choose any unitary operator u on $\mathcal{H}$ such that $\|u - 1\| < \varepsilon$ and $u\xi_n = \zeta$. (For example, choose u to be a rotation in the plane determined by ξ_n and ζ , and the identity on the orthogonal complement of this plane.) Define η by $\eta_k = u\xi_k$. Then $\|\eta_k - \xi_k\| < \varepsilon$ for all k so $\rho(\xi, \eta) < \varepsilon$, and clearly $\eta \in \mathbf{S}$. Since $\pi_n(\eta) = \zeta$, we have $B(\xi_n, \varepsilon) \subset \pi_n(B_\rho(\xi, \varepsilon) \cap \mathbf{S}) \subset \pi_n(U)$. Hence $\pi_n|_{\mathbf{S}}$ is open. $\square$

Now let D be a dense G_δ set in $\mathcal{H}_1$.

PROPOSITION 6. $\mathbf{S}(D) = \{\xi \in \mathbf{S} : \xi_n \in D \text{ for each } n\}$ is a dense G_δ in $\mathbf{S}$.

PROOF. Clearly $\mathbf{S}(D) = \bigcap_1^\infty \mathbf{S}_n(D)$ where $\mathbf{S}_n(D) = \{\xi \in \mathbf{S} : \xi_n \in D\}$. But $\mathbf{S}_n = (\pi_n|_{\mathbf{S}})^{-1}(D)$ is a dense G_δ in $\mathbf{S}$ by Lemmas 4 and 5. Since $\mathbf{S}$ is a complete metric space, a countable intersection of dense G_δ 's in $\mathbf{S}$ is again a dense G_δ in $\mathbf{S}$ [4, p. 98]. $\square$

Given $\xi \in \mathbf{S}$, for each positive integer n define $P_n(\xi)$ to be the projection onto the subspace spanned by $\{\xi_1, \xi_2, \dots, \xi_n\}$.

LEMMA 7. $\xi \in \mathbf{S}$ is an orthonormal basis if and only if for each $f \in \mathcal{H}_1$ there is an n such that $\|P_n(\xi)f\| > \frac{1}{2}$.

PROOF. If ξ is an orthonormal basis, then $P_n(\xi) \uparrow 1$ in the strong topology, so $\|P_n(\xi)f\|$ converges to $\|f\| = 1$. But if ξ is not an orthonormal basis, there is a vector f so that $P_n(\xi)f = 0$ for all n . $\square$

LEMMA 8. If $\{\xi_k\}$ is a sequence in $\mathbf{S}$ which converges to η , then for each n $P_n(\xi_k)$ converges in the norm topology to $P_n(\eta)$ as $k \rightarrow \infty$.

PROOF. Suppose the entries of ξ_k are denoted by ξ_{km} and those of η by η_m . Then for any $f \in \mathcal{H}_1$,

$$\begin{aligned} \|P_n(\xi_k)f - P_n(\eta)f\| &= \left\| \sum_{m=1}^n \langle \xi_{km}, f \rangle \xi_{km} - \sum_{m=1}^n \langle \eta_m, f \rangle \eta_m \right\| \\ &\leq \sum_{m=1}^n (|\langle \xi_{km} - \eta_m, f \rangle \xi_{km}| + |\langle \eta_m, f \rangle (\xi_{km} - \eta_m)|) \\ &\leq \sum_{m=1}^n 2\|\xi_{km} - \eta_m\| \\ &\leq 2n2^n \rho(\xi_k, \eta). \end{aligned}$$

Therefore $\|P_n(\xi_k) - P_n(\eta)\| \leq 2n2^n \rho(\xi_k, \eta)$. For each n , this converges to zero as $k \rightarrow \infty$. $\square$

Let $\mathbf{B} \subset \mathbf{S}$ be the set of all orthonormal bases.

PROPOSITION 9. $\mathbf{B}$ is a dense G_δ set in $\mathbf{S}$.

PROOF. Let $\xi \in \mathbf{S}$, and let $V_\varepsilon(\xi)$ be an ε -nbhd about ξ in the metric ρ . If N is chosen so that $2^N > 1/\varepsilon$, then any orthonormal set that differs from ξ only in entries with $n \geq N$ will belong to $V_\varepsilon(\xi)$. Clearly there are orthonormal bases that satisfy this condition. Therefore $\mathbf{B}$ is dense in $\mathbf{S}$.

Let $\{f_m\}$ be a countable dense set in $\mathcal{H}_1$. We claim that $\mathbf{B} = \bigcap_1^\infty \mathbf{O}_m$, where $\mathbf{O}_m = \{\xi \in \mathbf{S} : \|P_n(\xi)f_m\| > \frac{1}{2} \text{ for some } n\}$. To see this, note that by Lemma 7 if ξ is an orthonormal basis then ξ belongs to each $\mathbf{O}_m$. On the other hand, if ξ is not a basis then there is vector f such that $P_n(\xi)f = 0$ for all n . Choose m such that $\|f_m - f\| < \frac{1}{4}$. Then $\|P_n(\xi)f_m\| < \frac{1}{4}$ for all n , so ξ does not belong to $\mathbf{O}_m$.

We now show that each $\mathbf{O}_m$ is an open set in $\mathbf{S}$, which will complete the proof. Since $\mathbf{O}_m$ is the union over n of $\mathbf{O}_{mn} = \{\xi \in \mathbf{S} : \|P_n(\xi)f_m\| > \frac{1}{2}\}$, it is enough to show $\mathbf{O}_{mn}$ is open, or equivalently that its complement is closed. Suppose $\{\xi_k\}$ is a sequence in $\mathbf{S} \setminus \mathbf{O}_{mn}$ which converges to η , so that $\|P_n(\xi_k)f_m\| \leq \frac{1}{2}$ for each k . Since $P_n(\xi_k)$ converges to $P_n(\eta)$ in norm as $k \rightarrow \infty$ by Lemma 8, it follows that also $\|P_n(\eta)f_m\| \leq \frac{1}{2}$. Therefore $\eta \in \mathbf{S} \setminus \mathbf{O}_{mn}$, so $\mathbf{S} \setminus \mathbf{O}_{mn}$ is closed. $\square$

THEOREM 10. Let D be a dense G_δ set in the unit sphere of a separable Hilbert space $\mathcal{H}$. Then there is an orthonormal basis for $\mathcal{H}$ contained in D . In fact, the set of such orthonormal bases form a dense G_δ set in $\mathbf{S}(\mathcal{H})$.

PROOF. By Proposition 6, $\{\xi \in \mathbf{S} : \xi_i \in D \text{ for each } i\}$ is a dense G_δ in $\mathbf{S}$. By Proposition 9, $\mathbf{B}$ is a dense G_δ in $\mathbf{S}$. Since the intersection of two dense G_δ 's is again a dense G_δ , the theorem is proven. $\square$

REMARKS. 1. The fact that orthonormal bases form a dense G_δ in the space of orthonormal sequences is somewhat counterintuitive, since a dense G_δ is "large" in the sense that its complement is first category, while one might think that being a basis should be a rare property among orthonormal sequences. As an analogy one should consider the classic example of a dense G_δ of Lebesgue measure 0 in the unit interval [3, Theorem 1.6].

2. It is not hard to show that a set intersects every dense G_δ if and only if it is of second category. Thus all that is actually needed is to know that $\mathbf{S}(D) =$

$\{\xi \in \mathbf{S}: \xi_n \in D \text{ for each } n\}$ is second category in $\mathbf{S}$. However, there does not seem to be any convenient replacement for the condition that D be a dense G_δ that will guarantee this.

3. A dense subspace need not contain a dense G_δ in the unit sphere, since two dense subspaces may intersect in the zero vector whereas the intersection of two dense G_δ 's is again a dense G_δ . Therefore Theorem 10 is not a refinement of Proposition 1.

EXAMPLE. Let $\mathbf{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. Suppose that $\mathbf{M}$ has a cyclic and separating vector (i.e., $\mathbf{M}$ is *standard*). Then an orthonormal basis may be chosen for $\mathcal{H}$ consisting of vectors which are cyclic and separating for $\mathbf{M}$. This follows from [1] where it is shown that cyclic and separating vectors form a dense G_δ when they exist at all. The same result holds for vectors which are cyclic only or separating only.

This result can also be applied in the situation where one considers common cyclic and separating vectors for two, or even a countable number of von Neumann algebras, as in [2].

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