THE GROUP GENERATED BY UNIPOTENT OPERATORS¹

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ABSTRACT. The group generated by unipotent $n \times n$ complex matrices is $SL_n(C)$, and every member of the latter is a product of three unipotent matrices. The group generated by unipotent operators on Hilbert space $\mathcal{H}$ is $GL(\mathcal{H})$, and every invertible operator is a product of six unipotent operators of order 2.

An operator U on a Hilbert space is called unipotent [3, p. 100] if $U = 1 + N$, where N is nilpotent. It is called unipotent of order n if $N^n = 0$ and $N^{n-1} \neq 0$. Our aim is to characterize the group $\mathcal{G}$ generated by the unipotent operators and to show that every element of $\mathcal{G}$ is a product of a small number of unipotents. In a finite-dimensional space it is obvious that every operator in $\mathcal{G}$ has determinant one. We prove the converse of this. More precisely, we show that every operator with determinant 1 is a product of three unipotents. In the infinite-dimensional case we show that $\mathcal{G}$ is the group of all invertible operators and that every operator in $\mathcal{G}$ is a product of six unipotents of order 2.

We point out that other generators for the group of invertible operators were obtained by Radjavi [4].

The finite-dimensional case. In what follows $\mathcal{V}$ is an n -dimensional vector space over the complex field, where n is finite. The algebra of all linear transformations on $\mathcal{V}$ is denoted by $L(\mathcal{V})$. A linear transformation T on $\mathcal{V}$ is called cyclic if there exists a vector x such that the vectors $x, Tx, T^2x, \dots, T^{n-1}x$ span $\mathcal{V}$. It is well known that T is cyclic if and only if its Jordan canonical form contains only one Jordan block for every eigenvalue, and that this is the case if and only if the minimal polynomial of T equals its characteristic polynomial (see [2, Chapter 7]). As usual, the group of $n \times n$ complex matrices with determinant 1 is denoted by $SL_n(C)$.

LEMMA 1. *For each invertible $A \in L(\mathcal{V})$ with a singleton spectrum, there exists a unipotent U such that UA is cyclic and $\sigma(UA) = \sigma(A)$.*

PROOF. Let $\sigma(A) = \{\lambda\}$, and let $N = A - \lambda 1$. By considering the Jordan canonical form, we see that there exists a basis $\{e_1, e_2, \dots, e_n\}$ of $\mathcal{V}$ such that $Ne_j = \varepsilon_j e_{j+1}$ ($1 \leq j \leq n-1$) and $Ne_n = 0$, where $\varepsilon_j = 0$ or 1. Let S be the usual shift; i.e., $Se_j = e_{j+1}$ ($1 \leq j \leq n-1$) and $Se_n = 0$, and then let $J = S - N$. If $U = 1 + \lambda^{-1}J$,

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then U is unipotent and $UA = \lambda 1 + S - S^2 D$ for some transformation D which is diagonal with respect to the basis under consideration. Using the fact that $DS = SS^*DS$ and using induction, we get that for every positive integer m , $(UA - \lambda 1)^m = S^m + S^{m+1}R_m$ for some $R_m \in L(\mathcal{V})$. In particular, $(UA - \lambda 1)^{n-1} \neq 0$, but $(UA - \lambda 1)^n = 0$. This shows that $\sigma(UA) = \{\lambda\}$ and that UA is cyclic. $\square$

LEMMA 2. *For each invertible $A \in L(\mathcal{V})$, there exists a unipotent U such that UA is cyclic.*

PROOF. We may write A as a direct sum $A = A_1 \oplus \cdots \oplus A_k$, where $\sigma(A_j) = \{\lambda_j\}$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. By Lemma 1, there are unipotents U_j such that $U_j A_j$ is cyclic and $\sigma(U_j A_j) = \{\lambda_j\}$ for every j . If $U = U_1 \oplus \cdots \oplus U_k$, then UA is cyclic, since it is the direct sum of cyclic operators with disjoint spectra. $\square$

The following theorem may be of independent interest.

THEOREM 1. *Let A be a cyclic linear transformation, and let β_i and γ_i ($1 \leq i \leq n$) be complex numbers such that $\det A \prod_{i=1}^n \beta_i = \prod_{i=1}^n \gamma_i$. Then there exists a linear transformation B such that B has eigenvalues $\beta_1, \dots, \beta_n$ and BA has eigenvalues $\gamma_1, \dots, \gamma_n$. The eigenvalues are repeated according to algebraic multiplicity.*

PROOF. Let $f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n$ be the characteristic polynomial of A . By taking an appropriate basis, we may assume that the matrix of A is the companion matrix of $f(x)$ —i.e., the matrix

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & -a_0 \\ 1 & 0 & 0 & \cdots & -a_1 \\ 0 & 1 & 0 & \cdots & -a_2 \\ & & & \ddots & \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}.$$

We will show that there exist complex numbers $t_1, t_2, \dots, t_{n-1}$ such that if

$$D = \begin{bmatrix} \beta_1 & t_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & t_2 & \cdots & 0 \\ & & & \ddots & \\ & & & & t_{n-1} \\ 0 & 0 & \cdots & 0 & \beta_n \end{bmatrix},$$

then DC has eigenvalues $\gamma_1, \gamma_2, \dots, \gamma_n$.

The proof is by induction on n . There is nothing to prove for $n = 1$. Now suppose the above is true for $(n-1) \times (n-1)$ matrices ($n \geq 2$), and let C be as above. Let $g(x) = (x - \gamma_1) \cdots (x - \gamma_n)$. We can write $g(x)$ in the form $g(0) + xg_1(x)$ for a monic polynomial $g_1(x)$ of degree $n-1$. Let t_1 be a root of the polynomial equation $g_1(x) - a_1\beta_2 \cdots \beta_n = 0$. It follows that t_1 satisfies the equation $xg_1(x) - t_1 a_1 \beta_2 \cdots \beta_n = 0$, and so there exists a monic polynomial $h(x)$ of degree $n-1$

such that

$$xg_1(x) - t_1 a_1 \beta_2 \cdots \beta_n = (x - t_1)h(x).$$

It follows that $h(0) = a_1 \beta_2 \cdots \beta_n$, and so the product of the zeros of $h(x)$ is $(-1)^{n-1} a_1 \beta_2 \cdots \beta_n$. By the induction hypothesis, there exist complex numbers $t_2, \dots, t_{n-1}$ such that the characteristic polynomial of $D_0 C_0$ is $h(x)$, where C_0 is the companion matrix of the polynomial $a_1 + a_2 x + \cdots + a_{n-1} x^{n-1} + x^n$; and where D_0 is the matrix with $\beta_2, \dots, \beta_n$ on the main diagonal, $t_2, \dots, t_{n-1}$ on the diagonal above the main one, and zero everywhere else. Now let D be the $n \times n$ matrix with main diagonal $\beta_1, \dots, \beta_n$, with $t_1, \dots, t_{n-1}$ on the diagonal above, and with other entries zero. We have

$$DC = \left[\begin{array}{c|ccc} t_1 & 0 & \cdots & 0 & \mu \\ \hline \beta_2 & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{array} \begin{array}{c} \\ \\ \\ D_0 C_0 \\ \end{array} \right],$$

where $\mu = -(a_0 \beta_1 + a_1 t_1)$. By direct computation, we see that the characteristic polynomial of DC is

$$\begin{aligned} (x - t_1) \det(xI - D_0 C_0) - \mu \beta_2 \cdots \beta_n \\ = (x - t_1)h(x) + a_0 \prod_{i=1}^n \beta_i + t_1 a_1 \prod_{i=2}^n \beta_i \\ = xg_1(x) + a_0 \beta_1 \beta_2 \cdots \beta_n = g(x), \end{aligned}$$

so the eigenvalues of DC are $\gamma_1, \dots, \gamma_n$. This ends the induction proof. $\square$

The previous theorem implies that every cyclic A with determinant 1 is a product of two unipotents. Combining this with Lemmas 1 and 2, we have the following theorem.

THEOREM 2. *Every linear transformation with determinant 1 is a product of three unipotent transformations.*

The infinite-dimensional case. We consider a separable infinite-dimensional Hilbert space $\mathcal{H}$ over the complex field. The algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})$, and the group of all invertible operators is denoted by $GL(\mathcal{H})$. Let $\mathcal{U}$ be the set of all unipotent operators of order 2. The set of all products of n operators in $\mathcal{U}$ is denoted by $\mathcal{U}^n$. For an operator T , an equation such as $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ means that there is a unitary operator from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$ which carries T to the operator $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$.

We start with two lemmas.

LEMMA 3. *If both A and $1 + A$ are invertible, then $A \oplus A^{-1} \in \mathcal{U}^2$.*

PROOF. Let

$$F_1(A) = \begin{pmatrix} 2A(1+A)^{-1} & (A-1)(1+A)^{-1} \\ (1-A)(1+A)^{-1} & 2(1+A)^{-1} \end{pmatrix},$$

$$F_2(A) = \begin{pmatrix} 2A(1+A)^{-1} & (A^{-1}-1)(1+A)^{-1} \\ (A^2-A)(1+A)^{-1} & 2(1+A)^{-1} \end{pmatrix}.$$

Then both $F_1(A)$, $F_2(A)$ are in $\mathcal{U}$, and $A \oplus A^{-1} = F_1(A)F_2(A)$. $\square$

LEMMA 4. If $|\alpha| = 1$ and C is invertible, then $\alpha I \oplus C \in \mathcal{U}^4$.

PROOF. Choose $\varepsilon > 0$ such that $\varepsilon < 1$ and $\varepsilon\|C\| < 1$. Write $D = \varepsilon C$. Now $\alpha I \oplus C$ is unitarily equivalent to

$$\cdots \oplus \alpha I \oplus \alpha I \oplus \boxed{C} \oplus \alpha I \oplus \alpha I \oplus \cdots,$$

which is the product of the following two operators:

$$A = \cdots \oplus \alpha^4 D \oplus \bar{\alpha}^2 D^{-1} \oplus \alpha^2 D \oplus D^{-1} \oplus \boxed{D} \oplus \alpha \varepsilon^{-1} I \oplus \bar{\alpha} \varepsilon I \oplus \alpha^3 \varepsilon^{-1} I \oplus \cdots,$$

$$B = \cdots \oplus \bar{\alpha}^3 D^{-1} \oplus \alpha^3 D \oplus \bar{\alpha} D^{-1} \oplus \alpha D \oplus \boxed{\varepsilon^{-1} I} \oplus \varepsilon I \oplus \alpha^2 \varepsilon^{-1} I \oplus \bar{\alpha}^2 \varepsilon I \oplus \cdots.$$

Following the proof of Lemma 3, we have

$$\alpha^n D \oplus \alpha^{-n} D^{-1} = F_1(\alpha^n D) F_2(\alpha^n D),$$

$$\bar{\alpha}^n \varepsilon I \oplus \alpha^n \varepsilon^{-1} I = F_1(\bar{\alpha}^n \varepsilon I) F_2(\bar{\alpha}^n \varepsilon I).$$

Notice that $\sup_n \|F_1(\alpha^n D)\| < \infty$, $\sup_n \|F_2(\alpha^n D)\| < \infty$, $\sup_n \|F_1(\bar{\alpha}^n \varepsilon I)\| < \infty$ and $\sup_n \|F_2(\bar{\alpha}^n \varepsilon I)\| < \infty$. Thus A is unitarily equivalent to the product $A_1 A_2$, where

$$A_j = \cdots \oplus F_j(\alpha^4 D) \oplus F_j(\alpha^2 D) \oplus F_j(D) \oplus F_j(\bar{\alpha} \varepsilon I) \oplus F_j(\bar{\alpha}^3 \varepsilon I) \oplus \cdots$$

($j = 1, 2$). It follows from the proof of Lemma 1 that both A_1 and A_2 are in $\mathcal{U}$, and hence $A \in \mathcal{U}^2$. In the same way we have $B \in \mathcal{U}^2$. Therefore $\alpha I \oplus C \in \mathcal{U}^4$. $\square$

THEOREM 3. $\text{GL}(\mathcal{H}) = \mathcal{U}^6$.

PROOF. We consider two cases. In the first case, we assume that T is not a scalar modulo the compacts. By a theorem of Brown and Percy [1, Corollary 3.4], T is similar to an operator of the form $\begin{pmatrix} * & B \\ * & 1 \end{pmatrix}$. Notice that

$$\begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} * & B \\ * & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 1 \end{pmatrix},$$

where A is some invertible operator. (That A is invertible follows from the fact that T is invertible.) Also,

$$\begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ C & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

where $X = -CA^{-1}$. By Lemma 4, $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{U}^4$. Since T is similar to

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -X & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

we have $T \in \mathcal{U}^6$.

It now remains to consider the case $T = \lambda 1 + K$, where K is compact. We notice that $\lambda \neq 0$, since T is invertible. Let ε be a positive real number to be determined later. By Riesz theory, K is similar to $K_0 \oplus F$, where K_0 has spectral radius $< \varepsilon$, and where F is an operator on a finite-dimensional space. By a result of Rota [6, Theorem 2], K_0 is similar to an operator whose norm is $< \varepsilon$, so we may assume without loss of generality that $\|K_0\| < \varepsilon$. Write

$$K_0 = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix},$$

and let $A = (\lambda + K_1 + K_2)^{-1}(\alpha - \lambda - K_1)$, where $\alpha = 1$ if $\lambda \neq \pm 1$ and $\alpha = i$ if $\lambda = \pm 1$. (The operator $\lambda + K_1 + K_2$ is invertible if ε is small enough.) Let

$$S = \begin{pmatrix} 1 + A & -A \\ A & 1 - A \end{pmatrix}.$$

We have $(S - 1)^2 = 1$. Furthermore,

$$(\lambda + K_0)S = \begin{pmatrix} \alpha & \lambda - \alpha + L_1 \\ \alpha - \lambda + L_2 & 2\lambda - \alpha + L_3 \end{pmatrix},$$

where

$$L_1 = K_1 + K_2, \quad L_2 = (K_3 - K_1)(1 + A) + (K_4 - K_2)A$$

and

$$L_3 = K_1 + K_4 + (K_1 + K_2 - K_3 - K_4)A.$$

Let

$$R = \begin{pmatrix} 1 & 0 \\ \bar{\alpha}\lambda - 1 - \bar{\alpha}L_2 & 1 \end{pmatrix}.$$

Therefore

$$R(\lambda + K_0)S = \begin{pmatrix} \alpha & * \\ 0 & \bar{\alpha}\lambda^2 + J \end{pmatrix},$$

where

$$J = (1 - \lambda\bar{\alpha})(L_2 - L_1) + L_3 - \bar{\alpha}L_1L_2.$$

We notice that by our choice of α we have $\alpha \neq \bar{\alpha}\lambda^2$. From the expressions for J , L_1 , L_2 , L_3 we see that J is compact, and that we can choose ε small enough so that $\|J\| \leq |\alpha - \bar{\alpha}\lambda^2|$; so $\alpha \notin \sigma(\bar{\alpha}\lambda^2 + J)$. Now we have

$$R(\lambda + K_0)S = \begin{pmatrix} \alpha & * \\ 0 & C \end{pmatrix} \quad \text{with } \alpha \notin \sigma(C).$$

It follows by [5, Corollary 0.15] that $R(\lambda + K_0)S$ is similar to $\alpha 1 \oplus C$. Since

$$(R \oplus 1)[(\lambda + K_0) \oplus (\lambda + F)](S \oplus 1) = R(\lambda + K_0)S \oplus (\lambda + F),$$

and since T is similar to $(\lambda + K_0) \oplus (\lambda + F)$, we have $(R \oplus 1)T(S \oplus 1)$, similar to $\alpha 1 \oplus C \oplus (\lambda + F)$, which, by Lemma 4, belongs to $\mathcal{U}^4$. Since $R \oplus 1$ and $S \oplus 1$ are unipotent, we get $T \in \mathcal{U}^6$. This ends the proof of the theorem. $\square$

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