A THEOREM ON INFINITE NIELSEN EXTENSIONS

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ABSTRACT. It is shown that every Riemann surface with n punctures is the infinite Nielsen extension of a surface with n holes.

Let S_0 be a Riemann surface of genus g with n punctures and m holes. Assume that $6g - 6 + 2n + 3m > 0$ and that $m > 0$. Then S_0 can be represented as U/G where U is the upper half-plane and G is a torsion-free Fuchsian group of the second kind. There is an infinite set I of open intervals of $R \cup \{\infty\}$ on which G acts discontinuously. Let $\Omega = U \cup L \cup I$, where L denotes the lower half-plane. The quotient $S_0^d = \Omega/G$ is the *double* of S_0 . It is a compact surface, except for finitely many punctures. The surface S_0^d can be represented as U/G' , where G' is a torsion-free Fuchsian group of the first kind. Let D be a component of the preimage of S_0 under the natural projection $U \rightarrow U/G'$ and let G_1 be the stabilizer of D in G' . Then the surface $S_1 = U/G_1$ is the *Nielsen extension* of S_0 . Given a surface S_0 , let S_k be the Nielsen extension of S_{k-1} , for $k \in \mathbb{N}$. Define the *infinite Nielsen extension* of S_0 to be $S_\infty = S_0 \cup S_1 \cup S_2 \cup \dots$.

Bers has conjectured that given a Riemann surface S of type $(g; n, 0)$ and positive numbers $l_1, \dots, l_n$ there exists a surface of type $(g; 0, n)$ with boundary elements of lengths $l_1, \dots, l_n$ such that its infinite Nielsen extension is S (oral communication). This conjecture was proven in [4] for a torus with a hole, i.e. when $g = n = 1$. In this paper it will be proven for all g but without being able to specify the lengths of the holes. The Poincaré metric is used and the length of a boundary curve refers to the Poincaré length of the geodesic to which it is freely homotopic.

THEOREM. *Given a surface S of type $(g; n, 0)$ there exists a surface of type $(g; 0, n)$ such that its infinite Nielsen extension is S .*

The following inequalities concerning the lengths of geodesics of Nielsen extensions are needed in the proof. They are proven in [3].

I. Given a finite Riemann surface with a boundary curve of length l , then the length of the corresponding boundary curve of its Nielsen extension is less than $\frac{1}{2}l$. The length of the corresponding boundary element of its infinite Nielsen extension is therefore zero.

II. Let S be a finite Riemann surface such that all its boundary curves have length less than l . Assume that S has a simple closed geodesic of length x not homotopic to a boundary curve. Denote by x_n (x_∞) the length of the corresponding geodesic of its n th (its infinite) Nielsen extension. Then

$$cx < x_\infty < x \quad \text{and} \quad c_n x_n < x_\infty < x_n$$

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where

$$c = \prod_{i=1}^{\infty} \{1 - 2/\Pi \tan^{-1} 2 \sinh l/2^i\}$$

and

$$c_n = \prod_{i=n}^{\infty} \{1 - 2/\Pi \tan^{-1} 2 \sinh l/2^i\}.$$

Note that c and c_n are convergent infinite products which are close to 1 for small l and $c_n \rightarrow 1$ as $n \rightarrow \infty$.

The following lemma will be used in the proof of the theorem:

LEMMA. *For $k \geq 0$ and $n \geq 1$ let $\psi: R^k \times [0, \infty)^n \rightarrow R^k \times [0]^n$ be a continuous function such that its restriction to $R^k \times [0]^n$ is the identity map. Then the restriction of ψ to $R^k \times (0, \infty)^n$ is a surjection onto $R^k \times [0]^n$.*

To simplify notation, assume that the image space of ψ is R^k ; that is, $\psi: R^k \times [0, \infty)^n \rightarrow R^k$.

PROOF. Assume that the lemma is false, i.e. that there exists a point $P \in R^k \times [0]^n$ such that

$$\psi(R^k \times (0, \infty)^n) \subset Y = R^k - \{P\}.$$

Let $x \in R^k$ and $y \in [0, \infty)^n$ and define $\psi_y: R^k \rightarrow R^k$ by $\psi_y(x) = \psi(x, y)$. Let $y_0 = [0]^n$. Then, by the hypothesis of the lemma, $\psi_{y_0}(x) = x$ for all x . Pick $R > \|P\|$ and let

$$S = \{x \in R^k \mid \|x\| = R\}.$$

Then $\psi_{y_0}: S \rightarrow Y$ represents a nontrivial element of $H_{k-1}(Y; Z)$. For any point $y_1 \in (0, \infty)^n$, the map $\psi_{y_1}: S \rightarrow Y$ is homotopic to ψ_{y_0} and thus represents the same nontrivial element of $H_{k-1}(Y; Z)$. Since $H_{k-1}(R^k; Z) \simeq \{0\}$, $\psi_{y_1}: R^k \rightarrow Y$ induces the zero map

$$\psi_{y_1}^*: H_{k-1}(R^k; Z) \rightarrow H_{k-1}(Y; Z).$$

But ψ_{y_1} takes S to a nontrivial element of $H_{k-1}(Y; Z)$. This gives us a contradiction.

Given a topological orientable surface S of genus g with n boundary elements, the Fricke space F of S is the set of homotopy equivalence classes of pairs (h, R) where h is an orientation preserving homeomorphism of S onto a Riemann surface R .

THEOREM A [2]. *The Fricke space F is homeomorphic to $R^k \times [0, \infty)^n$, where $k = 6g - 6 + 2n$ and the last n coordinates represent the lengths of the holes.*

The topology on F is the weakest topology which makes lengths of closed geodesics continuous. Denote by x_C the length of a geodesic C .

THEOREM B [2]. *Let $f: F \rightarrow F$ be a function such that $x_C \circ f$ is continuous for every simple closed geodesic C . Then f is continuous.*

PROOF OF THE THEOREM. Let F denote the Fricke space of surfaces of genus g with n boundary elements. An element of F will refer to a point of $R^k \times [0, \infty)^n$ or to the marked surface represented by the point. Denote by F_0 the subset $R^k \times [0]^n$

of F and by F_1 the set $F - F_0$. The points of F_0 correspond to the surfaces with n punctures; the points of F_1 correspond to those surfaces with at least one boundary curve of positive length. The topology is the usual one on surfaces with holes.

Let $f_n: F_1 \rightarrow F_1$ be the map which takes a surface into its n th Nielsen extension. It will be shown to be continuous. Let $S \in F_1$ and $\{S_m\} \subset F_1$ such that $S_m \rightarrow S$. Then the doubles of S_m converge to the double of S , i.e. $S_m^d \rightarrow S^d$.

There exist Fuchsian groups $G_m = \langle A_1^m, \dots, A_{2\alpha}^m \rangle$ and $G = \langle A_1, \dots, A_{2\alpha} \rangle$ with $\alpha = 2g + n - 1$ such that

- (i) $S_m^d = U/G_m$ and $S^d = U/G$,
- (ii) $A_i^m \rightarrow A_i$ for $i = 1, \dots, 2\alpha$,
- (iii) the Nielsen extension of S_m is $U/\langle A_1^m, \dots, A_\alpha^m \rangle$, and
- (iv) the Nielsen extension of S is $U/\langle A_1, \dots, A_\alpha \rangle$.

But then $U/\langle A_1^m, \dots, A_\alpha^m \rangle$ converges to $U/\langle A_1, \dots, A_\alpha \rangle$ and $f_1(S_m) \rightarrow f_1(S)$. Therefore, f_1 is continuous. Since the composition of continuous functions is continuous, the function $f_n = f_1 \circ \dots \circ f_1$ is also continuous.

Define the function $f_\infty: F \rightarrow F_0$ as follows:

$$f_\infty(S) = \begin{cases} \lim_{n \rightarrow \infty} f_n(S) & \text{if } S \in F_1, \\ S & \text{if } S \in F_0. \end{cases}$$

Clearly, $f_n(S) = S_n \subset S_{n+1} = f_{n+1}(S)$ and $\lim_{n \rightarrow \infty} f_n(S) = S_0 \cup S_1 \cup S_2 \cup \dots = S_\infty$. Thus f_∞ maps a surface S of F_1 into its infinite Nielsen extension and it is the identity on F_0 . To prove the theorem it is necessary to show that

$$f_\infty(R^k \times (0, \infty)^n) = R^k \times [0]^n,$$

i.e. that every punctured surface of F_0 is the infinite Nielsen extension of some surface with n boundary curves of positive lengths.

It will now be shown that the function f_∞ is also continuous on F_1 . It is sufficient to show that given $\varepsilon > 0$ and $\delta > 0$, then for every simple closed geodesic of length x , $x \leq \delta$, there exists an N such that for all $n \geq N$, $|x_n - x_\infty| < \varepsilon$. Only geodesics not homotopic to boundary curves need to be considered. Suppose that ε and δ are given. Pick N such that $(1 - c_N)\delta < \varepsilon$ where c_N is the constant of inequality II. Then, for all $n \geq N$ and $x < \delta$,

$$|x_n - x_\infty| < (1 - c_N)\delta < \varepsilon.$$

To show that f_∞ is continuous everywhere, it is only necessary to show that if $S_m \rightarrow S_0$ where $\{S_m\} \subset F_1$ and $S_0 \in F_0$ then $f_\infty(S_m) \rightarrow f_\infty(S_0)$. This is sufficient because f_∞ is continuous on F_1 and is the identity on F_0 . Since $f_\infty(S_0) = S_0$, it must be shown that $f_\infty(S_m) \rightarrow S_0$. It is sufficient to show that, for m large, for every simple closed geodesic of S_m of length x_m , x_m is close to $(x_m)_\infty$. But since $S_m \rightarrow S_0$ and $S_0 \in F_0$, the lengths of the boundary curves of S_m must be very small for large m . Thus, by inequality II, x_m is close to $(x_m)_\infty$. Therefore, f_∞ is continuous on all of F and the function f satisfies the hypothesis of the lemma. But then $f(R^k \times (0, \infty)^n)$ is surjective onto $R^k \times [0]^n$ and the theorem is proved.

REMARK. After modifying the lemma slightly it can be shown that given a surface S of type $(g; n, 0)$ there exists an l such that there is a surface of genus g with n holes, each of length l , whose infinite Nielsen extension is S .

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REFERENCES

1. L. Bers, *Nielsen extensions of Riemann surfaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. **2** (1976), 17–22.
2. L. Bers and F. Gardiner, *A metric for Fricke space which is invariant under the modular group*, Adv. in Math. (to appear).
3. N. Halpern, *Some remarks on Nielsen extensions of Riemann surfaces*, Michigan Math. J. **30** (1983), 65–68.
4. —, *On the infinite Nielsen extension of a torus*, Complex Variables: Theory and Application **4** (1984), 111–117.

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