SOME PROPERTIES OF $\forall\exists$ MODELS IN THE ISOLS

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ABSTRACT. It is a consequence of theorems proved by Nerode [10] and Hirschfeld [7] that every countable model of $\forall\exists$ arithmetic is isomorphic to a subsemiring of a one-generator semiring of isols. We characterize, in terms of the generators of "Nerode semirings", the contents of *arbitrary* semirings $\mathbf{R}$ of isols that are models of $\forall\exists$ arithmetic, and we show that all such $\mathbf{R}$ are in fact models of the ω -true $\forall\exists$ sentences of *isol theory*. We solve one of the chief problems left open in [8], and in §3 we provide an example of the applied virtues of $\forall\exists$ -correct subsemirings of the isols.

1. Background. It was first shown by Nerode in [10] that there exist infinite isols X such that, for every recursive function $f: \omega \rightarrow \omega$, X belongs to the domain of the canonical extension f_Λ of f to isols. Such isols have been referred to as RST isols in the subsequent literature (see [5]). Nerode proved in [10] that these RST isols are precisely the ones that produce *recursive ultrapowers* when closed under all f_Λ ; i.e., if X is RST then the additively and multiplicatively closed set $\{f_\Lambda(X) \mid f \text{ recursive}\}$ of isols is a recursive ultrapower, and, conversely, every recursive ultrapower is (to within isomorphism) of this form. These facts are established in [10, Theorem 4.6]. Later work of Hirschfeld [7] linked the recursive ultrapowers to the countable models of $\forall\exists$ arithmetic. (Hirschfeld's definitions and theorems are formulated for Π_2^1 arithmetic; but, as he points out, Π_2^1 arithmetic is an inductive theory and hence, by the Chang-Los-Suszko theorem, is logically equivalent to $\forall\exists$ arithmetic.) Putting the two papers together yields a characterization of the countable models of $\forall\exists$ arithmetic (to be stated precisely in §2) that implies the following: Every countable model of $\forall\exists$ arithmetic is isomorphic to a subsemiring of a countable semiring $\mathbf{R}$ of isols, such that $\mathbf{R}$ = the closure of a singleton under the functions that are recursive in the sense of $\mathbf{R}$. (See [7] for the original meaning of "recursive in the sense of $\mathbf{R}$ "; we will show in §2 that it in fact means *being the canonical extension of a recursive function*.)

We denote, as usual, the semiring of all isols by Λ , and the collection of all regressive isols by Λ_R . We shall denote by $\Lambda(\forall\exists)$ the union of all semirings $\mathbf{R}$ of isols such that $\mathbf{R}$ is a model of $\forall\exists$ arithmetic, and by $\Lambda_R(\forall\exists)$ the union of all semirings $\mathbf{R}$ of *regressive* isols such that $\mathbf{R}$ is a model of $\forall\exists$ arithmetic. It is one of the general observations of §2 that $\Lambda(\forall\exists)$ is made up exclusively of isols that are either finite or are the generators of "Nerode semirings", i.e., generators of isolic semirings that are isomorphic to recursive ultrapowers.

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2. Generalities. The basic characterization theorem of Hirschfeld and Nerode is

LEMMA 1 (HIRSCHFELD, NERODE; SEE [7, 8, 10]). *The countable models of two-quantifier arithmetic = the existentially complete substructures of the recursive ultrapowers = the existentially complete subsemirings of the Nerode semirings. (By a Nerode semiring we mean the additive and multiplicative structure of the set $N(X) = \{f_\Lambda(X) | f \text{ recursive}\}$, where X is some RST isol. The symbol "=" in the statement of the lemma is, of course, to be understood in the sense of equality to within isomorphism.)*

Now, the discussion of recursive closure in models of $\forall\exists$ arithmetic provided in [7] makes it very natural to guess that the following lemma is true:

LEMMA 2. *Let $\mathbf{R}$ be any semiring of isol such that $\mathbf{R}$ is a model of $\forall\exists$ arithmetic. Then the functions that are recursive in the sense of $\mathbf{R}$, as defined in [7], are precisely the restriction to $\mathbf{R}$ of the canonical extensions to Λ of the ordinary recursive functions $f: \omega^k \rightarrow \omega$, $k \geq 1$.*

PROOF. Let $\mathbf{R}$ be as indicated, and let $f: \omega^k \rightarrow \omega$ be any k -place recursive function, $k \geq 1$. Then, by the Davis-Putnam-Robinson-Matijasevic (DPRM) theorem on diophantine representation, there are polynomials $P(y_1, \dots, y_t, x_1, \dots, x_k, z)$ and $Q(y_1, \dots, y_t, x_1, \dots, x_k, z)$ with natural number coefficients such that the following is a valid definition of f :

$$f(x_1, \dots, x_k) = z \leftrightarrow (\exists y_1, \dots, y_t)[P(y_1, \dots, y_t, x_1, \dots, x_k, z) \\ = Q(y_1, \dots, y_t, x_1, \dots, x_k, z)].$$

The right-hand side of this equivalence determines, according to the definition of [7], a function from $\mathbf{R}^k$ to $\mathbf{R}$ that is *recursive in the sense of $\mathbf{R}$* ; we shall continue to use the symbol " f " to denote the latter function. Conversely, let $\phi(x_1, \dots, x_k, z)$ be any Σ_1^0 formula defining, according to [7], a function recursive in the sense of $\mathbf{R}$. Then, by DPRM,

$$\phi(x_1, \dots, x_k, z) \leftrightarrow (\exists y_1, \dots, y_w)[P(y_1, \dots, y_w, x_1, \dots, x_k, z) \\ = Q(y_1, \dots, y_w, x_1, \dots, x_k, z)]$$

is valid for a suitable choice of polynomials P and Q with coefficients in ω . Thus, it suffices to consider only such existentially-quantified polynomial identities, in discussing the functions that are recursive in the sense of $\mathbf{R}$. (The statement

$$\forall \vec{x}[\Sigma_1^0(\vec{x}) \leftrightarrow \exists \vec{y}(P(\vec{y}, \vec{x}) = Q(\vec{y}, \vec{x}))]$$

breaks into two Π_2^0 -sentences. But recall our earlier remark about the proof in [7] of the equivalence between Π_2^0 arithmetic and $\forall\exists$ arithmetic.) But, for any recursive function $f: \omega^k \rightarrow \omega$ and any thus-associated P and Q , we have, by the Basic Nerode Metatheorem (BNM), the validity in Λ of the following universal Horn sentence (in the language L_N of isol theory and with f^+ , f^- chosen to be recursive combinatorial functions such that $f = f^+ - f^-$):

$$(*) \quad (\forall x_1, \dots, x_k, z)(\forall y_1, \dots, y_w)[P(y_1, \dots, y_w, x_1, \dots, x_k, z) \\ = Q(y_1, \dots, y_w, x_1, \dots, x_k, z) \rightarrow z + f^-(x_1, \dots, x_k) = f^+(x_1, \dots, x_k)].$$

Now if $X_1, \dots, X_k \in \mathbf{R}$, then there exist $Y_1, \dots, Y_w \in \mathbf{R}$ and a uniquely determined $Z \in \mathbf{R}$ so that $P(Y_1, \dots, Y_w, X_1, \dots, X_k, Z) = Q(Y_1, \dots, Y_w, X_1, \dots, X_k, Z)$; i.e., so that $f(X_1, \dots, X_k) = Z$ holds in Hirschfeld's sense in $\mathbf{R}$. But, by the validity of $(*)$ in Λ , this same polynomial identity implies that $Z = f_\Lambda(X_1, \dots, X_k)$. Thus, the function f in the sense of $\mathbf{R}$, as defined in [7], is just $f_\Lambda \upharpoonright_{\mathbf{R}}$, and the lemma is proved.

THEOREM 1. $\Lambda(\forall\exists) = \{X \in \Lambda \mid X \text{ is RST or finite}\}$.

PROOF. By Lemma 1, if X is RST or finite then $X \in \Lambda(\forall\exists)$. By Lemma 2, if $X \in \Lambda(\forall\exists)$ then, for every recursive function $f: \omega \rightarrow \omega$, $X \in \text{dom}(f_\Lambda)$; such an X is either finite or RST.

It follows from material in [10] that each Nerode semiring $\mathbf{N}(X)$ is isomorphic to a Nerode semiring $\mathbf{N}(X^*)$ in which the RST generator X^* is a *regressive* isol. By extension-theoretic generalities, not only X^* but in fact *all* elements of $\mathbf{N}(X^*)$ are regressive. Thus, there are plenty of $\forall\exists$ -correct semirings included in $\Lambda_R(\forall\exists)$. Restricting to the class Λ_R , we clearly have

THEOREM 1'. $\Lambda_R(\forall\exists) = \{X \in \Lambda_R \mid X \text{ is RST or finite}\} = A_R \cap \Lambda(\forall\exists)$.

We shall next establish that any *finitely generated* semiring $\mathbf{R}$ of isols that is a model of $\forall\exists$ arithmetic is actually a Nerode semiring (i.e., admits a single generator).

Let $\mathbf{M}$ be a model of $\forall\exists$ arithmetic. We shall say that $\mathbf{M}$ is *finitely generated* if $\mathbf{M}$ = the closure, under the functions recursive in the sense of $\mathbf{M}$, of some finite subset of $\mathbf{M}$.

LEMMA 3. *Let $\mathbf{R} \neq \omega$ be a finitely generated semiring of isols such that $\mathbf{R}$ is a model of $\forall\exists$ arithmetic. Then, there is an isol $X \in \mathbf{R}$ such that $\mathbf{R} = \mathbf{N}(X)$.*

PROOF. Let $X_1, \dots, X_k$ be a set of generators for $\mathbf{R}$. Since, by Lemma 2, f_Λ is defined on $\mathbf{R}$ and takes values in $\mathbf{R}$ for all recursive $f: \omega^k \rightarrow \omega$, we have in particular that $X_0 = 2^{X_1} 3^{X_2} \dots p_k^{X_k} \in \mathbf{R}$; here p_k , $k \geq 1$, denotes the k th prime element of ω , in order of magnitude. On the other hand, if $g_i: \omega \rightarrow \omega$ is the recursive function that maps n to $(n)_i$ = the highest power to which p_i divides n , then it is a routine consequence of the BNM that $(g_i)_\Lambda(X_0) = X_i$ for $1 \leq i \leq k$. Since $f_\Lambda(X_0)$ is defined and belongs to $\mathbf{R}$ for any recursive $f: \omega \rightarrow \omega$, we have $\mathbf{N}(X_0) \subseteq \mathbf{R}$. On the other hand, suppose $Y \in \mathbf{R}$. Then (again noting Lemma 2) it follows from the choice of $X_1, \dots, X_k$ and iterated use of Nerode's General Composition Theorem (GCT; see [9, Theorem 19.6]) that $Y = h_\Lambda(X_1, \dots, X_k)$ for some recursive $h: \omega^k \rightarrow \omega$. By one more use of the GCT, we get $Y = h_\Lambda((g_1)_\Lambda(X_0), \dots, (g_k)_\Lambda(X_0)) = f_\Lambda(X_0)$, where $f(x) = h(g_1(x), \dots, g_k(x))$. Thus $\mathbf{R} \subseteq \mathbf{N}(X_0)$, and we are done.

(Lemma 3, of course, is merely the reflection in Λ of a general property of models of $\forall\exists$ arithmetic. If we recall (see [7]) the theorem of Feferman, Scott and Tennenbaum that no nontrivial recursive ultrapower can model *three*-quantifier arithmetic, we get as an immediate corollary to Lemma 3 the fact that no nonstandard countable model of complete arithmetic can be finitely generated.)

It was (implicitly) observed in [8] that every Nerode semiring $\mathbf{N}(X)$ satisfies not only all true $\forall\exists$ statements of ordinary arithmetic but, in fact, all ω -true $\forall\exists$ sentences in the relatively rich language L_N of isol theory, containing a predicate symbol for each recursive relation and a function symbol for each almost recursive

combinatorial function. (These extra symbols are, of course, interpreted in Λ by the corresponding *canonical extensions*.)

With the aid of Lemma 3, we can extend this observation to arbitrary $\forall\exists$ -correct subsemirings of Λ :

THEOREM 2. *Let $\mathbf{R}$ be a semiring of isols that is a model of $\forall\exists$ arithmetic. Then $\mathbf{R}$ satisfies every ω -true $\forall\exists$ sentence ϕ of the extended language L_N .*

PROOF. Let $\phi = \forall x_1 \dots \forall x_k \exists y_1 \dots \exists y_t \psi$ be an $\forall\exists$ sentence of L_N such that $\omega \models \phi$. Let $X_1, \dots, X_k \in \mathbf{R}$. Let $\mathbf{R}'$ be the subsemiring of $\mathbf{R}$ generated by $X_1, \dots, X_k$; i.e., $\mathbf{R}'$ is the closure in $\mathbf{R}$ of $X_1, \dots, X_k$ under all functions recursive in the sense of $\mathbf{R}$. Then by [7, Theorem 1.8], $\mathbf{R}'$ is a model of $\forall\exists$ arithmetic. Now, by Lemma 2, the functions recursive in the sense of $\mathbf{R}$ are just the functions $f_\Lambda \upharpoonright_{\mathbf{R}}$, f recursive. Likewise, it is precisely the $f_\Lambda \upharpoonright_{\mathbf{R}'}$, f recursive, that are recursive in the sense of $\mathbf{R}'$. Thus, $\mathbf{R}'$ is the closure of its finite subset $\{X_1, \dots, X_k\}$ under the functions that are recursive in the sense of $\mathbf{R}'$, and so $\mathbf{R}'$ is finitely generated according to our definition of that notion. Therefore, by Lemma 3, there is an isol $X \in \mathbf{R}'$ such that $\mathbf{R}' = \mathbf{N}(X)$. (Indeed, we can explicitly describe X as $2^{X_1} \dots p_k^{X_k}$.) By [8, Theorem 2.7] (applied not just to “tame models” but, as it clearly can be, to arbitrary Nerode semirings), we have $\mathbf{N}(X) \models \phi$. So, since $X_1, \dots, X_k \in \mathbf{N}(X)$, there exist $Y_1, \dots, Y_t \in \mathbf{N}(X)$ such that $\mathbf{N}(X) \models \psi(X_1, \dots, X_k, Y_1, \dots, Y_t)$. Thus $\mathbf{R}' \models \psi(X_1, \dots, X_k, Y_1, \dots, Y_t)$, since $\mathbf{R}' = \mathbf{N}(X)$. But $\mathbf{R}' \subseteq \mathbf{R}$, whence

$$\mathbf{R} \models (\exists y_1) \dots (\exists y_t) \psi(X_1, \dots, X_k, y_1, \dots, y_t).$$

It follows that $\mathbf{R} \models \phi$, and we are done.

To conclude this section, we shall demonstrate that if $\mathbf{N}(X)$ is the special type of Nerode semiring called a *tame model* (see [1, 8]), then any nonstandard $\forall\exists$ -correct semiring $\mathbf{R}$ of isols that is embedded in $\mathbf{N}(X)$ must be embedded *cofinally* in $\mathbf{N}(X)$. We will show, as conjectured in [8], that not all regressively generated Nerode semirings are of this type. Recall, from [1 and 8], that $\mathbf{N}(X)$ is said to be a *tame model* if there is an infinite regressive isol $Y \in \mathbf{N}(X)$ such that $\mathbf{N}(X) = \{f_\Lambda(Y) \mid f: \omega \rightarrow \omega \text{ is a nondecreasing recursive function}\}$. Such a Y will be referred to as a *tame generator* for $\mathbf{N}(X)$. The existence of tame models is an elegant result due to Barback [1]. In what follows, we shall make the notational assumption, obviously with no loss of generality, that if $\mathbf{N}(X)$ is a tame model then X is a tame generator for it.

THEOREM 3. *Let $\mathbf{R}$ be a countable semiring of isols, $\mathbf{R} \neq \omega$, such that $\mathbf{R}$ is a model of $\forall\exists$ arithmetic, and let $\mathbf{N}(X)$ be a tame model such that $\mathbf{R}$ is a subsemiring of $\mathbf{N}(X)$. Then for any $Z \in \mathbf{N}(X)$, there is an isol $Y \in \mathbf{R}$ such that $Z \leq Y$ (i.e., such that $(\exists W)[Z + W = Y]$).*

PROOF. Let V be an infinite element of $\mathbf{R}$. Then $V = f_\Lambda(X)$ holds for some nondecreasing recursive function f such that $\text{ran}(f)$ is infinite. Since $\text{ran}(f)$ is infinite and recursive, we can define as follows a recursive function $h: \omega \rightarrow \omega$:

$$\begin{aligned} h(n) &= n, & n \notin \text{ran}(f); \\ h(n) &= \max\{m \mid f(m) \leq n\}, & n \in \text{ran}(f). \end{aligned}$$

By the BNM, then, the following sentence holds in Λ (where, for convenience, we refer *directly* to recursive functions, instead of indirectly via pairs of combinatorial

constituents):

$$(1) \quad (\forall x)(\forall v)[f(x) = v \rightarrow x \leq h(v)].$$

By Lemma 2, $\mathbf{R}$ is closed under the action of h_Λ ; hence $h_\Lambda(V) \in \mathbf{R}$. Since (see [8]) $\leq_\Lambda$ coincides with isolic $\leq$ in $\mathbf{N}(X)$, it follows from the truth of (1) in Λ that our theorem holds at least for the *generator* X of $\mathbf{N}(X): Z = X \leq h_\Lambda(V) \in \mathbf{R}$. Now let Z be an arbitrary element of $\mathbf{N}(X)$, and let $g: \omega \rightarrow \omega$ be a nondecreasing recursive function such that $Z = g_\Lambda(X)$. Since g is nondecreasing, the following (abbreviated) Horn sentence is true in ω and so also, by the BNM, in Λ :

$$(2) \quad (\forall x)(\forall w)[x \leq w \rightarrow g(x) \leq g(w)].$$

Hence, we see that since $X \leq W = h_\Lambda(V)$, we also have $g_\Lambda(X) = Z \leq g_\Lambda(h_\Lambda(V)) = (g \circ h)_\Lambda(V) \in \mathbf{R}$. The theorem is proved.

With the aid of Theorem 3 and results from [6 and 10], we can now settle one of the main questions raised in [8].

THEOREM 4. *There exist Nerode semirings, with regressive generators, that are not tame models.*

PROOF. Let $\mathbf{M}_1$ be a countable model of Peano arithmetic *and* of $\forall\exists$ arithmetic such that $\mathbf{M}_1$ is *not* a model of *complete* arithmetic. Let $\mathbf{N}(X_1)$ be a Nerode semiring such that $\mathbf{M}_1$ is isomorphically embedded in $\mathbf{N}(X_1)$. By [10, Theorem 5.1] (and its proof), there is a countable model $\mathbf{M}_2$ of complete arithmetic such that $\mathbf{N}(X_1)$ is isomorphically embeddable in $\mathbf{M}_2$ (It is alright here to use [10, Theorem 5.1], which does not directly embrace $\leq$ as a symbol of the formal language, for, in all models here under discussion, $x \leq y$ is correctly defined by $(\exists w)[x + w = y]$.) Finally, let $\mathbf{M}_2$ be isomorphically embedded in a Nerode semiring $\mathbf{N}(X_2)$, X_2 regressive. Identifying models with their various images under these embeddings, we have a chain $\mathbf{M}_1 \subseteq \mathbf{N}(X_1) \subseteq \mathbf{M}_2 \subseteq \mathbf{N}(X_2)$. Now, $\mathbf{M}_1$ cannot be a *cofinal* submodel of $\mathbf{M}_2$ in this chain, since by [6, Theorem 4] that would imply that $\mathbf{M}_1$ is an elementary submodel of $\mathbf{M}_2$, which it is not. If $\mathbf{M}_2$ is not cofinal in $\mathbf{N}(X_2)$, then, by Theorem 3, $\mathbf{N}(X_2)$ is not a tame model. But if $\mathbf{M}_2$ is cofinal in $\mathbf{N}(X_2)$, then $\mathbf{M}_1$ cannot be cofinal in $\mathbf{N}(X_2)$ lest it also be cofinal in $\mathbf{M}_2$. Hence, again, Theorem 3 implies $\mathbf{N}(X_2)$ is not tame. Thus $\mathbf{N}(X_2)$ is a witness to the truth of Theorem 4.

3. An example: realizing characteristics. Why consider $\forall\exists$ -correct subsemirings of Λ ? To the general model theorist, admittedly, even Lemma 1 may appear to be only a representational curiosity. From the point of view of isol theory, however, the benefits of looking at such semirings can be quite evident, in terms of simultaneous sharpening of results and streamlining of proofs. It seems possible, moreover, that this point of view could on occasion lead to some actual insights in the general study of "models of fragments of arithmetic", if only in the countable-universe case. We shall attempt to illustrate at least the purely isothoretic interest, by means of a concrete example.

Let $\{p_i\}_{i \geq 1}$ be the prime elements of ω , arranged in order of increasing size. Let $\{c_i\}_{i \geq 1}$ be a sequence, each term of which is either a natural number or the symbol ∞ . We say that $\{c_i\}$ is the *characteristic* of an isol X if (i) $c_i \in \omega \Rightarrow (p_i^{c_i} | X \ \& \ p_i^{c_i+1} \nmid X)$, and (ii) $c_1 = \infty \Rightarrow (\forall n \in \omega)[p_i^n | X]$. It was shown in [2] that

every $\{c_i\}$ is the characteristic of some isol. Later on, in [10], Nerode established an isoltheretic generalization of the Chinese Remainder Theorem (CRT) from which this result of [2] follows as an immediate corollary. (Ellentuck [4] carries the process further, exhibiting a way of coding, by means of a single isol, not just a given sequence of natural numbers but, much more generally, a *given sequence* $\{X_i\}$ of isols. That level of coding does not directly concern us here, but see our discussion of the second question posed in §4 below.)

We shall begin by noticing how much of Nerode's CRT can be realized within $\Lambda_R(\forall\exists)$. Verification of the following trivial lemma is left to the reader:

LEMMA 4. *Let $\mathbf{R}$ be any semiring of isols that is a model of $\forall\exists$ arithmetic, and let n be a positive integer. Then, for any $X \in \mathbf{R}$, there exist a $Y \in \mathbf{R}$ and a natural number $m < n$ such that $X = m + nY$.*

It is clear from Lemma 4 that no element of $\Lambda_R(\forall\exists)$ can be a witness to an instance of Nerode's CRT in which (see [10]) X is to leave no finite remainder on division by some prime power p^{k+1} . On the other hand, all "normal" instances of the generalized CRT are witnessed in $\Lambda_R(\forall\exists)$:

LEMMA 5. *To each prime $p \in \omega$, let there be assigned a sequence $\{a_{p,j}\}_{j \in \omega}$ such that $(\forall j)[a_{p,j} \in [0, 1, \dots, p-1]]$. Then there is an isol $X \in \Lambda_R(\forall\exists)$ such that for all primes p and all $j \in \omega$, X leaves remainder $a_{p,0} + a_{p,1}p + \dots + a_{p,j}p^j$ on division by p^{j+1} .*

PROOF. The ordinary Chinese Remainder Theorem and the Henkin Consistency Lemma combine to provide a countable model $\mathbf{M}$ of $\forall\exists$ arithmetic and a nonstandard element ξ of $\mathbf{M}$ such that ξ leaves the indicated remainders in $\mathbf{M}$. Lemma 1 produces an isomorphic copy of $\mathbf{M}$ sitting inside a Nerode semiring $\mathbf{N}(X)$, and, as noted in §2, it can be assumed that X is regressive and hence that $\mathbf{N}(X) \subseteq \Lambda_R(\forall\exists)$.

Now the coding of characteristics in $\Lambda_R(\forall\exists)$ follows:

THEOREM 5 (cf. [2, THEOREM 1]). *Let $\{c_i\}_{i \geq 1}$ be any sequence each of whose terms is a natural number or the symbol ∞ . Then there is an isol $X \in \Lambda_R(\forall\exists)$ such that $\{c_i\}$ = the characteristic of X .*

PROOF. Obtain X from Lemma 5, taking $a_{p_i,t} = 0$ for all $t \in \omega$ if $c_i = \infty$ and, if $c_i \in \omega$, taking $a_{p_i,t} = 0$ for $t < c_i$ and $a_{p_i,t} = 1$ for $t \geq c_i$. Theorem 105(a) of [3] (together with its proof) ensures that $p_i^{c_i+1} \nmid X$ in the second alternative.

4. Some questions. It might seem tempting to guess that there must be a lot of uncountable models of $\forall\exists$ arithmetic representable as subsemirings of Λ , along with the countable ones. There are, however, some severe restrictions on this possibility, suggesting that in fact there may not be *any* such models:

THEOREM 6. (i) $\Lambda(\forall\exists)$ does not include any semiring $\mathbf{R}$ such that $\mathbf{R}$ is a model of $\forall\exists$ arithmetic and $\mathbf{R}$ has a proper initial segment $\mathbf{R}'$ with $\text{card}(\mathbf{R}') \geq \aleph_1$. (Hence, by the MacDowell-Specker theorem, every model $\mathbf{M}$ of complete arithmetic of cardinality $\aleph_1$ has an elementary end-extension $\mathbf{M}^*$ of cardinality $\aleph_1$ such that $\mathbf{M}^*$ is not embeddable in the isols.)

(ii) $\Lambda_R(\forall\exists)$ does not include any uncountable semiring $\mathbf{R}$ that is a model of $\forall\exists$ arithmetic.

PROOF. (i) If such an $\mathbf{R}$ existed, at least one of its elements would have to have more than $\aleph_0$ predecessors, which of course is not possible in the isols. (ii) follows from [9, Theorem 16.13].

This suggests

Question 1. Are there any models of $\forall\exists$ arithmetic of cardinality $\aleph_1$ that are embeddable in the isols?

We shall hazard a guess that the answer is “no”. The vague idea behind this conjecture has three components: First, our suspicion that the answer to Question 2 below is affirmative; second, the fact that every Nerode semiring is isomorphic to a Nerode semiring with a regressive generator, and that this isomorphism can be achieved without changing ultrafilters; and third, Theorem 6(ii).

Question 2. Must every countable semiring of isols that models $\forall\exists$ arithmetic be not merely isomorphic to, but literally *equal* to a subsemiring of some Nerode semiring?

To prove that the answer to Question 2 is “yes”, we would appear to need something like a version for RST isols of the general coding theorem of [4]. There seems no way to take *direct* advantage of [4], however, since the method there depends rather heavily on the availability of a certain first-order definition of ω as a subset of Λ . That definition fails utterly to distinguish ω from the rest of a given $\forall\exists$ -correct semiring $\mathbf{R} \subseteq \Lambda$; and any predicate that *would* so define ω must have a universal closure of level $\forall\exists\forall$ or higher. Nevertheless, we venture to guess that the answer to Question 2 is, indeed, “yes”.

ADDED IN PROOF (February 4, 1986). Three further remarks seem in order.

(1) Our proof of Theorem 4 (or any of various other arguments deriving Theorem 4 from Theorem 3) actually shows that Nerode semirings need not be *embeddable in tame models*.

(2) Lemma 3 (along with its proof) is just a counterpart, for *arbitrary* countable models of $\forall\exists$ arithmetic, to an observation about *existentially complete* models made in Chapter 9 of the Hirschfeld-Wheeler monograph *Forcing, Arithmetic, Division Rings* (Lecture Notes in Math., vol. 454, Springer-Verlag, 1975).

(3) The answer to Q1, if negative, cannot be negative for purely order-theoretic reasons. For, by a result of Ellentuck's (*Uncountable suborderings of the isols*, Compositio Math. **26** (1973)), Λ embeds (with respect to the isolic predecessor relation) any linear order of cardinality $\aleph_1$ in which each element has only countably many predecessors. So, the question is whether this can sometimes be done in an arithmetic-preserving manner.

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