ON THE FORMAL GROUPS APPROACH TO COBORDISM

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ABSTRACT. We give a direct proof of the fact that the complex bordism ring U_* is a polynomial ring, based on Quillen's approach. The proof is self-contained, apart from the assumption that the groups U_n are all finitely generated and an algebraic lemma on formal groups.

1. Preliminaries. *U-structures.* A *U-structure* on a map $f: X \rightarrow Y$ between (para-compact, smooth) manifolds is a homotopy class of complex structures on $\nu_X \oplus f^*\tau_Y$ where ν_X is the normal bundle of X in $\mathbf{R}^N$ (suitable N) and τ_Y is the tangent bundle of Y . When $Y = pt$ we have a *U-structure* on X , which may also be specified by a complex structure on $\tau_X \oplus \varepsilon$ (ε a real trivial bundle of suitable dimension). A *U-manifold* is a manifold with *U-structure* and a *U-map* is a map with *U-structure*. Clearly,

- (i) the composite of two *U-maps* is a *U-map*,
- (ii) any map between *U-manifolds* is a *U-map*, in the obvious way,
- (iii) any map homotopic to a *U-map* is a *U-map*.

Bordism and cobordism. For a pair of spaces X, A , bordism $\Omega_*(X, A) = \bigoplus \Omega_n(X, A)$ is defined as usual by compact *U-manifolds* modulo *U-bordism*. If $f: X, A \rightarrow Y, B$ is a map we define f_* by $f_*[g] = [f \circ g]$ where $g: M, \partial M \rightarrow X, A$.

Let X be a manifold. Define $U_n(X)$ by *U-maps* $M \rightarrow X$ (where $\dim M = n$ and M is closed (= compact, boundaryless)) modulo *U-bordism* of *U-maps*. Thus, for X closed, $U_*(X)$ is the usual complex cobordism (but graded by dimension) and, for X a *U-manifold*, $U_*(X) = \Omega_*(X)$, in the obvious way. If $f: X \rightarrow Y$ is a *U-map* then $f_*: U_*(X) \rightarrow U_*(Y)$ may be defined by $f_*[g] = [f \circ g]$. If $h: X \rightarrow Y$ is a proper smooth map, define $h^*: U_*(Y) \rightarrow U_*(X)$ as follows. If M is closed and $g: M \rightarrow Y$ a *U-map* we may assume (modulo homotopy) that g is transverse to h and consider the pull-back diagram

$$\begin{array}{ccc} R & \xrightarrow{h_1} & M \\ \downarrow g_1 & & \downarrow g \\ X & \xrightarrow{h} & Y. \end{array}$$

R is closed and g_1 is a *U-map*. Define $h^*[g] = [g_1]$ (independent of choice of g).

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- LEMMA 1. (i) For U -maps, $(f \circ g)_* = f_* \circ g_*$.
 (ii) For proper maps, $(h \circ k)^* = k^* \circ h^*$.
 (iii) f_* is invariant under homotopy.
 (iv) If $h: X \rightarrow Y$ is proper and $g: W \rightarrow Y$ is a U -map with h, g transverse, then, in the pull-back diagram,

$$\begin{array}{ccc} R & \xrightarrow{h_1} & W \\ \downarrow g_1 & & \downarrow g \\ X & \xrightarrow{h} & Y, \end{array}$$

h_1 is proper, g_1 is a U -map and $h^* \circ g_* = g_{1*} \circ h_1^*$. ■

We write $U_* = U_*(pt) = \Omega_*(pt)$. $U_*(X)$ is a U_* -algebra (with unit 1_X if X is closed). If $f: X \rightarrow Y$ is proper then f^* is a U_* -algebra map (with $f^*1_Y = 1_X$ if X, Y closed).

If E is a complex bundle over a closed manifold X then the zero-section $i: X \rightarrow E$ is a U -map. The Euler class $e(E)$ is $i^*i_*1_X$ in $U_*(X)$. We have $e(E_1 \oplus E_2) = e(E_1) \cdot e(E_2)$ and $e(f^*E) = f^*e(E)$.

Let $\zeta = S^{2m+1} \times_{S^1} \mathbb{C}$ be the dual of the canonical line-bundle over $CP(m)$ and let $z = e(\zeta)$. An easy argument, involving the Thom isomorphism, shows that

$$U_*(CP(m)) \simeq U_*[t]/t^{m+1} = 0$$

(isomorphism of U_* -algebras taking t to z).

Lens-spaces. Let $p \geq 2$ and $L = S^{2m+1}/\Gamma$ where $\Gamma(v) = \exp(2\pi i/p)v$. If X is a manifold with smooth $\mathbb{Z}/p$ -action T , write $L \times X$ for $S^{2m+1} \times_{\mathbb{Z}/p} X$, where the action on $S^{2m+1} \times X$ is $\Gamma \times T$. $L \times X$ fibers over L with fiber X .

Suppose $X \rightarrow Y$ is an equivariant embedding (X, Y manifolds with smooth $\mathbb{Z}/p$ -actions) whose normal bundle has an equivariant complex structure. Then the induced embedding $L \times X \rightarrow L \times Y$ has a complex structure on its normal bundle and is thus a U -map.

L is a U -manifold via the equivariant complex structure on $\tau_{S^{2m+1}} \oplus \mathbb{R}$ coming from $\mathbb{C}^{m+1}$. If θ is a complex linear action of $\mathbb{Z}/p$ (on a vector space V , say) we write $L \times \theta$ for the complex bundle $L \times V$. For example, if ω acts on $\mathbb{C}$ by $\omega(z) = \exp(2\pi i/p)z$, then $L \times \omega$ is the dual of the canonical line bundle. Clearly, $L \times (\omega^j) \simeq (L \times \omega)^j$ (j -fold $\otimes$). Let ρ be the action $\rho(z_1, z_2, \dots, z_p) = (z_2, \dots, z_p, z_1)$ on $\mathbb{C}^p$. Then $\rho = \text{diagonal} \oplus \nabla$ where $\nabla \simeq \omega \oplus \omega^2 \oplus \dots \oplus \omega^{p-1}$, and $L \times \nabla \simeq q^*\Sigma$ where $\Sigma = \zeta \oplus \zeta^2 \oplus \dots \oplus \zeta^{p-1}$ and $q: L \rightarrow CP(m)$ is the quotient map. We may identify q with the projection of the circle-bundle associated to ζ^p . Then the following piece of the Gysin sequence

$$\dots \rightarrow U_*(CP(m)) \xrightarrow{e(\zeta^p)} U_*(CP(m)) \xrightarrow{q^*} U_*(L) \rightarrow \dots$$

is exact at the middle term.

Splitting principle. Let $T(n) =$ diagonal matrices in $U(n)$ and $BT(n) \rightarrow BU(n)$ be induced by $T(n) \subseteq U(n)$.

LEMMA 2. $\Omega_*(BT(n)) \rightarrow \Omega_*(BU(n))$ is onto.

PROOF. The proof follows from the following well-known facts.

(a) $BU(n)$ has a C.W. decomposition in which every cell is a cycle,

(b) $H_*(BT(n); \mathbf{Z}) \rightarrow H_*(BU(n); \mathbf{Z})$ is onto,

(c) $\Omega_*(BT(n)) \rightarrow H_*(BT(n); \mathbf{Z})$ is onto,

together with an induction up the skeletons (taking $BT(n) \rightarrow BU(n)$ to be cellular).

■

Now a C^n -bundle E over a closed U -manifold defines $[E] \in \Omega_*(BT(n))$. From Lemma 2 we deduce that (splitting principle) every element of $\Omega_*(BU(n))$ may be written as $[E]$ where E is a $\oplus$ of complex line bundles.

Formal group law. There exist $a_{ij} \in U_*$ (for all $i, j > 0$) such that, for any two complex line bundles L_1, L_2 over a closed manifold,

$$e(L_1 \otimes L_2) = e(L_1) + e(L_2) + \sum a_{ij} e(L_1)^i e(L_2)^j.$$

If, in $U_*(X)$, we write $u \sim v$ to mean that $u - v \in \sum a_{ij} U_*(X)$ (i.e., $u - v$ is a sum of elements each of which is divisible by at least one a_{ij}), then we have

$$e(L_1 \otimes L_2) \sim e(L_1) + e(L_2).$$

The main result we will prove (in §2) is

THEOREM (QUILLEN). $1 \in U_0$ and the a_{ij} generate U_* as a ring.

Characteristic numbers. To get relations in U_* we will use the splitting principle together with certain cobordism characteristic numbers.

Define $\Phi: \Omega_n(BU(n)) \rightarrow U_n(CP((p-1)n))$ (a close relative of the Steenrod-tom Dieck p th power operation) by the formula $\Phi(E) = \pi_* e(\Sigma \otimes E)$ where E is a bundle over M , $\pi: P \times M \rightarrow P$ is projection, $P = CP((p-1)n)$, $\otimes$ is external $\otimes$, $\Sigma = \zeta \oplus \cdots \oplus \zeta^{p-1}$ and $p \geq 2$.

LEMMA 3. Φ is well defined (i.e., bordism invariant).

PROOF. Let B be a U -bordism between M and N (say), $i: M \rightarrow B$, $j: N \rightarrow B$ inclusions and E a bundle over B . Then, in $U_*(P \times B)$, we have

$$(1 \times i)_* e(\Sigma \otimes i^* E) = (1 \times j)_* e(\Sigma \otimes j^* E),$$

by relative transversality. Applying π_* we get the lemma. ■

DEFINITION. Define $\phi: \Omega_n \rightarrow U_n (= \Omega_n)$ by $\phi(M) =$ coefficient of $z^{(p-1)n}$ in $\Phi(\nu_M)$ where ν_M is the complex normal bundle of the n -dimensional U -manifold M in (for convenience) $\mathbf{R}^{3n}$. By Lemma 3, ϕ is well defined (since $[M] = [N]$ implies $[\nu_M] = [\nu_N]$).

LEMMA 4. $\phi(M) \sim s[M]$ where $s = (p-1)!^n$.

PROOF. By the splitting principle we have $[\nu_M] = [L_1 \oplus \cdots \oplus L_n]$ in $\Omega_n(BU(n))$ where $L_1, \dots, L_n$ are complex line bundles over N , say. Of course, $[N] = [M]$.

Writing $\times$ for external product in $U_*(P \times N)$ we have

$$\begin{aligned} e(\Sigma \otimes (L_1 \oplus \cdots \oplus L_n)) &\sim \prod_{k,l} (kz \times 1_N + 1_P \times e(L_l)) \\ &= s(z^{(p-1)n} \times 1_N) + \text{lower powers of } z. \end{aligned}$$

Applying π_* and taking coefficients of $z^{(p-1)n}$ gives the lemma. ■

(Note $s = (p-1)!$ since $1 \leq k \leq p-1$ and $1 \leq l \leq n$.)

PROPOSITION 1. *If $n > 0$ then $\phi(M) \sim$ an element of pU_* .*

PROOF. Since $U_1 = 0$ we may take $n \geq 2$. Write $V = \mathbf{R}^{3n}$ and take the normal bundle E of M to be embedded in V . Thus $M \subset E \subset V$. In the pullback diagram

$$\begin{array}{ccc} L \times M & \xrightarrow{\pi} & L \\ \downarrow q \times 1 & & \downarrow q \\ P \times M & \xrightarrow{\pi} & P \end{array}$$

we have $q^* \circ \pi_* = \pi_* \circ (q \times 1)^*$, by Lemma 1. Thus

$$\begin{aligned} q^* \Phi(E) &= q^* \pi_* e(\Sigma \otimes E) = \pi_*(q \times 1)^* e(\Sigma \otimes E) \\ &= \pi_* e(q^* \Sigma \otimes E) = \pi_* e((L \times \nabla) \otimes E). \end{aligned}$$

We claim that $q^* \Phi(E) = 0$. Granted that $q^* \Phi(E) = 0$, it follows, from the Gysin sequence for ζ^p , that $\Phi(E)$ is a multiple of $e(\zeta^p)$ in $U_*(P)$ and hence that $\Phi(E) \sim$ an element of $pU_*(P)$. Comparing coefficients of $z^{(p-1)n}$, we deduce that $\phi(M) \sim$ an element of pU_* , which proves the proposition.

To show that $q^* \Phi(E) = 0$ it suffices to show that $j_* e((L \times \nabla) \otimes E) = 0$, where $j: L \times M \rightarrow L \times V$ (and $J: L \times M^p \rightarrow L \times V^p$ below) are embeddings induced by $M \subset V$. We may identify $(L \times \nabla) \otimes E$ with $L \times (\nabla \otimes E)$ since both are quotients of $(S \times C^{p-1}) \otimes E = S \times (C^{p-1} \otimes E)$ by the same relation. We get the commutative diagram

$$\begin{array}{ccccc} & L \times M & \xrightarrow{\delta} & L \times M^p & \\ & \downarrow i & & \downarrow \epsilon & \\ L \times M & \xrightarrow{i} & L \times (\nabla \otimes E) & & \\ \downarrow j_1 & & \downarrow j_3 & & \\ L \times E & \xrightarrow{i_1} & L \times (\rho \otimes E) & \xrightarrow{\delta_1} & L \times E^p \\ \downarrow j_2 & & & & \downarrow \gamma \\ L \times V & \xrightarrow{\Delta} & & & L \times V^p \end{array}$$

(Curved arrows $j: L \times M \rightarrow L \times V$ and $J: L \times M^p \rightarrow L \times V^p$ are also shown.)

in which M^p etc. stand for $M \times \cdots \times M$ etc. with actions $(m_1, m_2, \dots, m_p) \rightarrow (m_2, \dots, m_p, m_1)$ etc. and all maps are induced by the obvious inclusions, zero-sections or diagonal maps.

Clearly all horizontal maps are proper and all vertical maps are U -maps. Let us consider the three rectangles:

$$(1) \quad \begin{array}{ccc} \cdot & \xrightarrow{\delta} & \cdot \\ \downarrow i & & \downarrow \epsilon \\ \cdot & \xrightarrow{\delta_1} & \cdot \\ \downarrow j_3 & & \downarrow \end{array} \quad (2) \quad \begin{array}{ccc} \cdot & \xrightarrow{i} & \cdot \\ \downarrow j_1 & & \downarrow j_3 \\ \cdot & \xrightarrow{i_1} & \cdot \end{array} \quad (3) \quad \begin{array}{ccc} \cdot & \xrightarrow{i_1} & \cdot \\ \downarrow j_2 & & \downarrow \gamma \\ \cdot & \xrightarrow{\Delta} & \cdot \end{array}$$

We claim that the above rectangles are pull-back diagrams. As for (1) consider $L \times E^p$. This is a bundle over $L \times M^p$. Its restriction to the diagonal $L \times M$ is $L \times (E \oplus \cdots \oplus E)$, i.e. $L \times (\rho \otimes E)$. Clearly, in $L \times E^p$ the zero section $L \times M^p$ and (the total space of) the bundle $L \times (\rho \otimes E)$ intersect transversally in the zero section of the bundle $L \times (\rho \otimes E)$, which is $L \times M$. This proves that (1) is a pull-back diagram. As for (2), it is enough to observe that the bundle $L \times (\rho \otimes E)$ over $L \times M$ splits into the Whitney sum of the “diagonal” bundle $L \times E$ and the “antidiagonal” bundle $L \times (\nabla \otimes E)$. Finally, (3) is clearly a pull-back diagram. Applying Lemma 1 repeatedly we have the identity

$$j_* i_* i_* 1_{L \times M} = \Delta^* J_* 1_{L \times M^p}.$$

In fact,

$$\begin{aligned} j_* i_* i_* 1_{L \times M} &= j_* i_* i_* \delta^* 1_{L \times M^p} \\ &= j_{2*} (j_{1*} i^*) i_* \delta^* 1_{L \times M^p} \\ &= j_{2*} (i_1^* j_{3*}) i_* \delta^* 1_{L \times M^p} \quad (\text{by Lemma 1}) \\ &= j_{2*} i_1^* (j_3^*) i_* \delta^* 1_{L \times M^p} \\ &= j_{2*} i_1^* \delta_1^* \epsilon_* 1_{L \times M^p} \quad (\text{by Lemma 1}) \\ &= j_{2*} (\delta_1 i_1)^* \epsilon_* 1_{L \times M^p} \\ &= \Delta^* \gamma_* \epsilon_* 1_{L \times M^p} \quad (\text{by Lemma 1}) \\ &= \Delta^* (\gamma \epsilon)_* 1_{L \times M^p} = \Delta^* J_* 1_{L \times M^p}. \end{aligned}$$

Now J is homotopic to the composite of projection $L \times M^p \rightarrow L$ and inclusion $L \rightarrow L \times V^p$. But $\dim(L \times V^p) - \dim L - \dim(L \times V) = (p-1)n - 1$ is > 0 (since we took $p, n \geq 2$). Thus $\Delta^* J_* = 0$ and the proposition is proved. ■

2. Proof of the theorem. Let $n > 0$ and $p \geq 2$ be prime. Then $\phi(M) \sim$ an element of pU_n (Proposition 1) and $\phi(M) \sim s[M]$, where $s = (p-1)!$ (Lemma 4). Therefore, $[M] \sim$ an element of pU_n .

Let C_* be the ideal $\sum a_{ij}U_*$ of U_* . For all $n \geq 0$, U_n/C_n is finitely generated since U_n is. We showed above that for $n > 0$ any prime $p \geq 2$ divides any element of U_n/C_n . Thus, $U_n/C_n = 0$ for $n > 0$.

Since no a_{ij} lies in $U_0 = \mathbf{Z}$ we may repeatedly substitute $U_0 + C_*$ for U_* in the definition $C_* = \sum a_{ij}U_*$ to deduce that the a_{ij} generate $\bigoplus_{n>0} U_n$ as a ring. This proves the theorem. ■

3. Polynomial structure of U_* . In §2 we saw that $U_* = \text{subring } L$ generated by $1 \in U_0$ and the a_{ij} . A calculation involving the defining properties of formal groups [K] shows that there is a (ring) epimorphism $\mathbf{Z}[t_2, t_4, t_6, \dots] \rightarrow L$ (taking t_{2i} into U_{2i}) and hence an epimorphism $\mathbf{Z}[t_2, t_4, t_6, \dots] \rightarrow U_*$. But, by a well-known Chern-numbers calculation [MS], the rank of U_{2r} is at least the number of partitions of r . Thus we have a ring isomorphism $\mathbf{Z}[t_2, t_4, t_6, \dots] \simeq U_*$. ■

The unoriented analogue of Proposition 1 is proved similarly. The logarithm for unoriented bordism N_* then provides an epimorphism $(\mathbf{Z}/2)[t_2, t_4, t_5, t_6, \dots] \rightarrow N_*$ (where $t_i \rightarrow N_i$ ($i \neq 1, 3, 7, \dots$)) and a Stiefel-Whitney numbers calculation [MS] shows this to be an isomorphism.

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