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ON THE UNIVALENT FUNCTIONS STARLIKE WITH RESPECT TO A BOUNDARY POINT

PAVEL G. TODOROV

ABSTRACT. For the examined functions, we have obtained a structure formula and estimates for $|f(z)/(1-z)|$ and $|\arg(f(z)/(1-z))|$, the moduli of the partial sums of the coefficient series and the moduli of the coefficients.

Recently, Robertson [**1**] introduced the following two classes of univalent functions:

DEFINITION 1. Let G^* denote the class of functions $f(z)$ analytic in $D = \{z| |z| < 1\}$, normalized so that $f(0) = 1$, $f(1) = \lim_{r\to 1} f(r) = 0$, and such that for some real α, $\operatorname{Re}[e^{i\alpha} f(z)] > 0$, $z \in D$. In addition let $f(z)$ map D univalently on a domain starlike with respect to $f(1)$. Let the constant function 1 also belong to the class G^*.

DEFINITION 2. Let G denote the class of functions $f(z)$ analytic and nonvanishing in D, normalized so that $f(0) = 1$ and such that

$$(1)\qquad \operatorname{Re}\left\{\frac{2zf'(z)}{f(z)} + \frac{1+z}{1-z}\right\} > 0 \qquad (z \in D).$$

Robertson [**1**] conjectured that the classes G^* and G coincide. Recently, Lyzzaik [**2**] proved this conjecture.

Now we shall continue the study of the class G.

THEOREM 1. *The function $f(z)$ belongs to the class G if and only if $f(z)$ can be written in the form*

$$(2)\qquad f(z) = (1-z)\exp\left\{-\int_{-\pi}^{\pi} \ln(1 - ze^{-it})\,d\mu(t)\right\} \qquad (z \in D),$$

for some probability measure μ defined on the interval $[-\pi, \pi]$.

PROOF. It follows from (1) and a classic result due to Herglotz that

$$\frac{2zf'(z)}{f(z)} + \frac{1+z}{1-z} = \int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\,d\mu(t)$$

holds in D for some probability measure $\mu(t)$. A simple integration now yields the desired structure formula (2) for $f(z)$.

THEOREM 2. *For a fixed $z \in D$, we have the relation*

$$(3)\qquad \{w|w = (1-z)/f(z),\ f(z) \in G\} = \{w| |w-1| \leqq |z|\},$$

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where for $z \neq 0$ the equality holds only for the functions

$$f(z) = \frac{1-z}{1-e^{i\omega}z} = 1 + \sum_{n=1}^{\infty} [e^{in\omega} - e^{i(n-1)\omega}]z^n \in G, \qquad \omega \in [-\pi, \pi]. \tag{4}$$

PROOF. From (2) it follows that

$$\ln \frac{1-z}{f(z)} = \int_{-\pi}^{\pi} \ln(1 - ze^{-it}) d\mu(t) \tag{5}$$

holds in D. According to the Carathéodory principle [**3**] (see also [**4**, p. 543; **5**]), from (5) it follows that for a fixed $z \in D$ we shall have the relation

$$\left\{ \varsigma | \varsigma = \ln \frac{1-z}{f(z)},\ f(z) \in G \right\} = \mathrm{CH}\{\varsigma | \varsigma = \ln(1 - ze^{-it}),\ t \in [-\pi, \pi]\}, \tag{6}$$

where CH denotes the convex hull of the set in the braces. The function

$$\varsigma = \ln w \tag{7}$$

maps the w-plane out along the negative real axis onto the strip $\{\varsigma | -\pi < \operatorname{Im} \varsigma < \pi\}$. It is clear geometrically that the function (7) maps the disc $\{w \,|\, |w-1| \leqq |z|,\ |z| < 1\}$ onto some convex domain lying in this strip. Therefore, for a fixed $z \in D$ from (6) the relation

$$\left\{ w | w = \frac{1-z}{f(z)},\ f(z) \in G \right\} = \mathrm{CH}\{w | w = 1 - ze^{-it},\ t \in [-\pi, \pi]\} \tag{8}$$

follows. Now the relation (8) can be written as (3) with (4).

COROLLARY 2.1. *We have the relation*

$$\bigcup_{f \in G} \left\{ w | w = \frac{1-z}{f(z)},\ z \in D \right\} = \{w \,|\, |w-1| < 1\}. \tag{9}$$

PROOF. The relation (9) follows from (3) for $|z| \to 1$.

COROLLARY 2.2. *For $z \in D$ and $f(z) \in G$, we have the sharp estimates*

$$1/(1+|z|) \leqq |f(z)/(1-z)| \leqq 1/(1-|z|) \tag{10}$$

and

$$|\arg(f(z)/(1-z))| \leqq \arcsin|z|, \tag{11}$$

where for $z \neq 0$ equality holds only for the functions (4) *at the "critical points"*

$$z = \pm|z|e^{-i\omega} \tag{12}$$

and

$$z = |z|e^{\pm i(\pi/2 \mp \omega - \arcsin|z|)}, \tag{13}$$

respectively.

PROOF. The inequalities (10) with (12) and (11) with (13) follow from (3) on the basis of the inequalities $1 - |z| \leqq |w| \leqq 1 + |z|$ and $|\arg w| \leqq \arcsin|z|$, respectively.

COROLLARY 2.3. *For each function*

$$f(z) = 1 + d_1 z + d_2 z^2 + \cdots + d_n z^n + \cdots \tag{14}$$

of the class G in D, the inequalities

$$|1 + d_1 + d_2 + \cdots + d_n| \leqq 1, \qquad n = 1, 2, \ldots, \tag{15}$$

and

$$|d_n| \leqq 2, \qquad n = 1, 2, \ldots, \tag{16}$$

hold with equality in (15) *only for the functions* (4) *with* $\omega \in [-\pi, \pi]$ *and in* (16) *only for the function* (4) *with* $\omega = \pm\pi$, *i.e.,* $f(z) = (1 - z)/(1 + z)$.

PROOF. We write $w = f(z)/(1 - z)$. Then (9) yields $|1/w - 1| < 1$, i.e., $\operatorname{Re} w > \frac{1}{2}$. If

$$w = \sum_{n=0}^{\infty} S_n z^n, \qquad S_n = d_0 + d_1 + \cdots + d_n,\ d_0 = 1,$$

the Borel-Carathéodory inequalities applied to $2w - 1$ yield $2|S_n| \leqq 2$, as required.

REMARK. The results in this paper can also be obtained by other methods and results due to Robertson [**1**, p. 331, Theorem 1; **6**, p. 318, Theorem 7; **7**, pp. 385–386], Schild [**8**] and Pinchuk [**9**, pp. 722, 727–728, 732].

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DEPARTMENT OF MATHEMATICS, PAISSII HILENDARSKI UNIVERSITY, 4000 PLOVDIV, BULGARIA