WEAK CONVERGENCE OF MEASURES AND WEAK TYPE $(1, q)$ OF MAXIMAL CONVOLUTION OPERATORS

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ABSTRACT. Let G^* be the maximal convolution operator associated with a sequence of L^1 kernels. We show that if G^* is of weak type $(1, q)$, $1 \leq q < \infty$, over a subset $\mathcal{N}$ of $\mathcal{M}$ (the space of all finite positive Borel measures on $\mathbf{R}^h$ endowed with the weak topology), then G^* is of weak type $(1, q)$ over the closed cone in $\mathcal{M}$ generated by $\mathcal{N}$. As a particular case we obtain a well-known result by de Guzman.

This paper is concerned with the question of making easier the task of finding out about the weak type $(1, q)$ of a maximal convolution operator. We give a general criterion which reduces to a small set the study of the action of such an operator in establishing its weak type $(1, q)$, $1 \leq q < \infty$. Previous results in this direction have been obtained by K. H. Moon [4] and mainly by M. de Guzman [3, Chapter 4] (see also M. T. Carrillo and M. de Guzman [2]). In particular, de Guzman [3, Theorem 4.1.1] has proved that a maximal convolution operator is of weak type $(1, 1)$ if and only if it is of weak type $(1, 1)$ over finite sums of Dirac deltas. We refer the reader to a recent note by H. Carlsson [1] for a nice application of this kind of result.

As the title suggests, our criterion is related to the so-called “weak” topology (for the space of all finite Borel measures on $\mathbf{R}^h$). Also, it gives back, as easy corollaries, the results by Moon and de Guzman, making clear (perhaps) their topological background and somewhat simplifying the proofs. As a by-product the technical lemma on which our criterion is based shows also that the sufficient condition for the weak type (p, p) , $p > 1$, given by Carrillo and de Guzman [2] is satisfied only if all kernels are zero. We also show by an example that in our result the “weak” topology cannot be replaced by the “vague” one.

To begin let us remark that the technique we follow makes it convenient to consider the maximal convolution operators as defined on a space of Borel measures more than, as usual, on a space of functions.

Throughout we will denote by $\mathcal{M}$ the space of all finite positive Borel measures on $\mathbf{R}^h$ endowed with the “weak” topology, i.e. the topology for which a net $\{\mu_\alpha\}$ in $\mathcal{M}$ converges to $\mu \in \mathcal{M}$ (written $\mu_\alpha \xrightarrow{w} \mu$) if and only if

$$\lim_{\alpha} \int_{\mathbf{R}^h} \varphi(x) \mu_\alpha(dx) = \int_{\mathbf{R}^h} \varphi(x) \mu(dx)$$

for every bounded continuous function $\varphi: \mathbf{R}^h \rightarrow \mathbf{R}$. To avoid trivialities we will not consider the zero measure as an element of $\mathcal{M}$. Also, for any $\mu \in \mathcal{M}$, we will set $\|\mu\| = \mu(\mathbf{R}^h)$. It is well known that $\mathcal{M}$ is metrizable. For this and other basic

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facts about the weak topology we refer the reader to the fundamental work by V. S. Varadarajan [6] or to any good textbook in probability and measure theory.

We will denote by L^1 the usual $L^1(\mathbf{R}^h)$ space with respect to the Lebesgue measure m and by $\|\cdot\|_1$ the usual norm in L^1 . Obviously, if $f \in L^1$, $f \geq 0$, and μ is the measure with density f with respect to m , then $\mu \in \mathcal{M}$ and $\|\mu\| = \|f\|_1$. Given a function $g \in L^1$ (kernel), its convolution $g * \mu$ with a measure $\mu \in \mathcal{M}$ is the element of L^1 defined by

$$g * \mu(x) = \int_{\mathbf{R}^h} g(x-y)\mu(dy).$$

Let $(g_i)_{i \in I}$ be a countable collection of kernels. Then the maximal convolution operator $G^* \mu$ associated with $(g_i)_{i \in I}$ is defined, for every $\mu \in \mathcal{M}$, by

$$G^* \mu(x) = \sup_{i \in I} |g_i * \mu(x)|.$$

Given a set $\mathcal{N} \subset \mathcal{M}$ we say that G^* is of weak type $(1, q)$, $1 \leq q < \infty$, over $\mathcal{N}$ provided that there exists a constant c such that

$$m\{G^* \nu > \lambda\} \leq (c\|\nu\|/\lambda)^q$$

for every $\lambda > 0$ and $\nu \in \mathcal{N}$, where $\{G^* \nu > \lambda\}$ stands for $\{x \in \mathbf{R}^h : G^* \nu(x) > \lambda\}$. If $\mathcal{N} = \mathcal{M}$ we will simply say that G^* is of weak type $(1, q)$. Obviously, if G^* is of weak type $(1, q)$ according to our definition, it is of weak type $(1, q)$ in the standard sense (i.e., over functions in L^1). As our result will show the converse is also true.

We need one more notation. Given $\mathcal{N} \subset \mathcal{M}$ we denote by $\widehat{\mathcal{N}}$ the minimal closed cone in $\mathcal{M}$ containing $\mathcal{N}$. (By a cone we mean a subset $\mathcal{C}$ of $\mathcal{M}$ such that $\mu \in \mathcal{C}$ implies $r\mu \in \mathcal{C}$ for every $r > 0$.) In other words, $\widehat{\mathcal{N}}$ is the closure in $\mathcal{M}$ of $\{r\nu : \nu \in \mathcal{N}, r > 0\}$.

We are now in a position to state our main result.

THEOREM 1. *Let $(g_i)_{i \in I}$ be any countable collection of kernels in L^1 and let G^* denote the associated maximal convolution operator. Assume that G^* is of weak type $(1, q)$ over $\mathcal{N} \subset \mathcal{M}$. Then G^* is of weak type $(1, q)$ over $\widehat{\mathcal{N}}$.*

PROOF. For every maximal convolution operator G^* we can define a mapping

$$(1) \quad \mu \rightarrow c(\mu; G^*),$$

from $\mathcal{M}$ to $[0, \infty]$, by putting

$$(2) \quad c(\mu; G^*) = \sup_{\lambda > 0} \frac{\lambda}{\|\mu\|} (m\{G^* \mu > \lambda\})^{1/q}.$$

We easily have that the mapping (1) satisfies

$$(3) \quad c(r\mu; G^*) = c(\mu; G^*)$$

for every $\mu \in \mathcal{M}$ and $r > 0$. Also it is clear that G^* is of weak type $(1, q)$ over a set $\mathcal{N} \subset \mathcal{M}$ if and only if $\sup_{\nu \in \mathcal{N}} c(\nu; G^*) < \infty$. Then the theorem is an immediate consequence of the following lemma.

LEMMA. Let G^* be the maximal convolution operator associated with a countable collection $(g_i)_{i \in I}$ of kernels in L^1 . Then $\mu \rightarrow c(\mu; G^*)$ is a lower semicontinuous mapping from $\mathcal{M}$ to $[0, \infty]$.

PROOF. For every $\mu \in \mathcal{M}$ we have

$$\begin{aligned} c(\mu; G^*) &= \sup_{\lambda > 0} \sup_J \frac{\lambda}{\|\mu\|} (m\{G_J^* \mu > \lambda\})^{1/q} \\ &= \sup_J \sup_{\lambda > 0} \frac{\lambda}{\|\mu\|} (m\{G_J^* \mu > \lambda\})^{1/q} = \sup_J c(\mu; G_J^*), \end{aligned}$$

where J ranges over all finite subsets of I and G_J^* denotes the maximal convolution operator associated with $(g_i)_{i \in J}$. As a consequence of this it can be assumed that I itself is a finite set, say $I = \{1, \dots, k\}$. Also, since $\mu_\alpha \xrightarrow{w} \mu$ implies $\mu_\alpha / \|\mu_\alpha\| \xrightarrow{w} \mu / \|\mu\|$ and owing to (3), it will be enough to show that the restriction of the mapping (1) to the set $\mathcal{M}_1 = \{\mu \in \mathcal{M} : \|\mu\| = 1\}$ of all Borel probability measures on $\mathbf{R}^h$ is lower semicontinuous.

To this aim let $\{\mu_n\}$ be any sequence in $\mathcal{M}_1$ such that $\mu_n \xrightarrow{w} \mu \in \mathcal{M}_1$ as $n \rightarrow \infty$. We will show that $\liminf_{n \rightarrow \infty} c(\mu_n; G^*) \geq c(\mu; G^*)$.

We first notice that for every $n \in N$ we have

$$G^* \mu(x) \leq G^* \mu_n(x) + \sup_{i \in I} |g_i * \mu(x) - g_i * \mu_n(x)|$$

almost everywhere in $\mathbf{R}^h$.

Now let γ be any number satisfying $\gamma < c(\mu; G^*)$ and let $\bar{\lambda} > 0$ be fixed such that

$$\gamma < (\bar{\lambda} / \|\mu\|) (m\{G^* \mu > \bar{\lambda}\})^{1/q} = \bar{\lambda} (m\{G^* \mu > \bar{\lambda}\})^{1/q}.$$

According to the above remark, for every α , $0 < \alpha < \bar{\lambda}$, and every $n \in N$, we have

$$\begin{aligned} (4) \quad \gamma &< \bar{\lambda} (m\{G^* \mu > \bar{\lambda}\})^{1/q} \\ &\leq \bar{\lambda} \left(m\{G^* \mu_n > \bar{\lambda} - \alpha\} + m \left\{ \sup_{i \in I} |g_i * \mu - g_i * \mu_n| > \alpha \right\} \right)^{1/q} \\ &\leq \bar{\lambda} (m\{G^* \mu_n > \bar{\lambda} - \alpha\})^{1/q} + \bar{\lambda} \left(\sum_{i=1}^k m\{|g_i * \mu - g_i * \mu_n| > \alpha\} \right)^{1/q} \\ &\leq \frac{\bar{\lambda}}{\bar{\lambda} - \alpha} c(\mu_n; G^*) + \bar{\lambda} \left(\sum_{i=1}^k m\{|g_i * \mu - g_i * \mu_n| > \alpha\} \right)^{1/q}. \end{aligned}$$

Next let ε be any positive number and let $\varphi_i: \mathbf{R}^h \rightarrow \mathbf{R}$, $i = 1, \dots, k$, be uniformly continuous, bounded functions in L^1 such that $\|g_i - \varphi_i\|_1 < \varepsilon$, $i = 1, \dots, k$. Then,

for every $i = 1, \dots, k$, and every $n \in \mathbb{N}$, by Chebyshev's inequality we have

$$\begin{aligned}
 & m\{|g_i * \mu - g_i * \mu_n| > \alpha\} \\
 & \leq m\{|(g_i - \varphi_i) * \mu| > \alpha/3\} + m\{|\varphi_i * \mu - \varphi_i * \mu_n| > \alpha/3\} \\
 & \quad + m\{|(\varphi_i - g_i) * \mu_n| > \alpha/3\} \\
 & \leq (3/\alpha)\|(g_i - \varphi_i) * \mu\|_1 + m\{|\varphi_i * \mu - \varphi_i * \mu_n| > \alpha/3\} \\
 & \quad + (3/\alpha)\|(\varphi_i - g_i) * \mu_n\|_1 \\
 & \leq (6/\alpha)\|g_i - \varphi_i\|_1 + m\{|\varphi_i * \mu - \varphi_i * \mu_n| > \alpha/3\} \\
 & < 6\varepsilon/\alpha + m\{|\varphi_i * \mu - \varphi_i * \mu_n| > \alpha/3\}.
 \end{aligned}$$

On the other hand, for every $i = 1, \dots, k$, the family of functions $\{\psi_{i,x}: x \in \mathbb{R}^h\}$, given by $\psi_{i,x}(y) = \varphi_i(x - y)$ for every $y \in \mathbb{R}^h$ and every $x \in \mathbb{R}^h$, is equicontinuous and uniformly bounded. Hence, by a well-known result due to R. Ranga Rao [5, Theorem 3.1],

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^h} \left| \int_{\mathbb{R}^h} \psi_{i,x}(y) \mu_n(dy) - \int_{\mathbb{R}^h} \psi_{i,x}(y) \mu(dy) \right| = 0,$$

i.e. $\varphi_i * \mu_n(x) \rightarrow \varphi_i * \mu(x)$ uniformly in $\mathbb{R}^h$, and so $\{|\varphi_i * \mu - \varphi_i * \mu_n| > \alpha/3\} = \emptyset$ for n large enough. It follows that, for n large enough,

$$(5) \quad \left(\sum_{i=1}^k m\{|g_i * \mu - g_i * \mu_n| > \alpha\} \right)^{1/q} < \left(\frac{6k\varepsilon}{\alpha} \right)^{1/q}.$$

After all that, by (4) and (5), we get

$$\begin{aligned}
 \gamma & \leq \liminf_{n \rightarrow \infty} \left[\frac{\bar{\lambda}}{\bar{\lambda} - \alpha} c(\mu_n; G^*) + \bar{\lambda} \left(\sum_{i=1}^k m\{|g_i * \mu - g_i * \mu_n| > \alpha\} \right)^{1/q} \right] \\
 & \leq \frac{\bar{\lambda}}{\bar{\lambda} - \alpha} \liminf_{n \rightarrow \infty} c(\mu_n; G^*) + \bar{\lambda} \limsup_{n \rightarrow \infty} \left(\sum_{i=1}^k m\{|g_i * \mu - g_i * \mu_n| > \alpha\} \right)^{1/q} \\
 & \leq \frac{\bar{\lambda}}{\bar{\lambda} - \alpha} \liminf_{n \rightarrow \infty} c(\mu_n; G^*) + \bar{\lambda} \left(\frac{6k\varepsilon}{\alpha} \right)^{1/q}.
 \end{aligned}$$

Then, letting, in turn, $\varepsilon \rightarrow 0$, $\alpha \rightarrow 0$ and $\gamma \rightarrow c(\mu; G^*)$, we obtain the desired inequality $c(\mu; G^*) \leq \liminf_{n \rightarrow \infty} c(\mu_n; G^*)$. This concludes the proof of the lemma.

We now give some easy consequences of our Theorem 1.

COROLLARY 1. *Let G^* be the maximal convolution operator associated with a countable collection $(g_i)_{i \in I}$ of kernels in L^1 . Assume that G^* is of weak type $(1, q)$ over the set of all finite sums of Dirac deltas concentrated at different points of $\mathbb{R}^h$. then G^* is of weak type $(1, q)$ (over $\mathcal{M}$).*

PROOF. Let $\mathcal{N}$ denote the set of all finite sums of Dirac deltas (concentrated at different points). We will prove the corollary showing that $\widehat{\mathcal{N}} = \mathcal{M}$.

To begin with, consider any measure $\mu \in \mathcal{M}$ of the form $\mu = k\delta_{x_0}$ where $k \in \mathbb{N}$ and δ_{x_0} is the Dirac delta at x_0 . Then $\mu \in \mathcal{N}^\wedge$ because there exists a sequence $\{\nu_n\}$ in $\mathcal{N}$ weakly converging to μ (take ν_n to be the sum $\sum_{i=1}^k \delta_{x_{i,n}}$, where $x_{i,n} \rightarrow x_0$, as $n \rightarrow \infty$, for $i = 1, \dots, k$). In the same way any linear combination of Dirac deltas with positive integer coefficients belongs to $\mathcal{N}^\wedge$. Since $\mathcal{N}^\wedge$ is a cone it is possible to take any positive rational as a coefficient in the above mentioned combination and it is well known that the set of these combinations is dense in $\mathcal{M}$. The corollary hence follows.

COROLLARY 2. *Let G^* be as in Corollary 1. Assume that G^* is of weak type $(1, q)$ over the set of all measures having as density (with respect to m) the characteristic function of a set E , with $m(E) < \infty$. Then G^* is of weak type $(1, q)$.*

PROOF. Let $\mathcal{N}$ denote the set of measures in the assumption of the corollary. Then $\mathcal{N}^\wedge$ contains all finite sums of Dirac deltas concentrated at different points. Hence, by Theorem 1 and Corollary 1, the conclusion is obtained.

It is now clear that Corollaries 1 and 2 imply (and in fact slightly extend) de Guzman's Theorem 4.1.1 in [3]. Also Moon's result [4] is covered by Corollary 2. Finally, let us explicitly remark that, owing to Corollary 2, a maximal convolution operator is of weak type $(1, q)$ according to our definition if and only if it is in the usual sense (i.e. over L^1).

The following example shows that in Theorem 1 (and hence in the Lemma) the role of the weak topology cannot be played by the "vague" topology (i.e. the topology on $\mathcal{M}$ for which a net $\{\mu_\alpha\}$ converges to μ if and only if

$$\lim_{\alpha} \int_{\mathbb{R}^h} \varphi(x) \mu_\alpha(dx) = \int_{\mathbb{R}^h} \varphi(x) \mu(dx)$$

for every continuous and compactly supported function $\varphi: \mathbb{R}^h \rightarrow \mathbb{R}$). For the sake of brevity we will give the explicit calculations only in the case $q = 1$.

EXAMPLE. *(A maximal convolution operator which is of weak type $(1, 1)$ over a vaguely dense subset of $\mathcal{M}$ but not over $\mathcal{M}$.)*

Let $h = 1$ and let the kernels $(g_i)_{i \in \mathbb{N}}$ be defined by $g_i(x) = 1/|x|$ if $1/i \leq |x| \leq i$ and $g_i(x) = 0$ otherwise. Then, using the monotone convergence theorem, it is easy to see that the associated maximal operator G^* is given by

$$G^* \mu(x) = \int_{\mathbb{R}} g(x-y) \mu(dy),$$

for every $\mu \in \mathcal{M}$, where $g(x) = 1/|x|$ if $x \neq 0$ and $g(0) = 0$. It follows that G^* is not of weak type $(1, 1)$ (take as μ the measure whose density is the characteristic function of a bounded interval; then $c(\mu; G^*) = \infty$). Moreover, for any Dirac delta δ_{x_0} , we have

$$m\{G^* \delta_{x_0} > \lambda\} = 2/\lambda$$

for every $\lambda > 0$. Hence, for any linear combination of Dirac deltas $\mu = \sum_{j=1}^k \beta_j \delta_{x_j}$, we have

$$m\{G^* \mu > \lambda\} \leq \sum_{j=1}^k m\left\{G^* \delta_{x_j} > \frac{\lambda}{k\beta_j}\right\} = \frac{2k}{\lambda} \sum_{j=1}^k \beta_j = \frac{2k}{\lambda} \|\mu\|$$

for every $\lambda > 0$, and so $c(\mu; G^*) \leq 2k < \infty$.

Now, let $\{\mu_n\}$ be a sequence whose terms are all measures of the form $\sum_{j=1}^k \beta_j \delta_{x_j}$ with $\beta_1, \dots, \beta_k, x_1, \dots, x_k$ rationals, $\beta_1, \dots, \beta_k > 0$. The set of such measures is well known to be (weakly and hence) vaguely dense in $\mathcal{M}$. Then it is easily established that, if $\{y_n\}$ is any sequence in $\mathbf{R}$ with $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$ and $\{\gamma_n\}$ is any sequence of positive numbers, then also $\mathcal{V} = \{\mu_n + \gamma_n \delta_{y_n} : n \in \mathbf{N}\}$ is a vaguely dense subset of $\mathcal{M}$.

To conclude the example we show that the γ_n 's can be chosen in such a way that G^* is of weak type $(1, 1)$ over $\mathcal{V}$. Indeed, if we take $\gamma_n \geq 2c(\mu_n; G^*)\|\mu_n\|$, $n \in \mathbf{N}$, then for every $\nu = \mu_n + \gamma_n \delta_{y_n} \in \mathcal{V}$ and every $\lambda > 0$, we have

$$\begin{aligned} m\{G^* \nu > \lambda\} &\leq m\left\{G^* \mu_n > \frac{\lambda}{2}\right\} + m\left\{G^* \delta_{y_n} > \frac{\lambda}{2\gamma_n}\right\} \\ &\leq \frac{2c(\mu_n; G^*)\|\mu_n\|}{\lambda} + \frac{4\gamma_n}{\lambda} \leq \frac{5\gamma_n}{\lambda} < \frac{5}{\lambda}\|\nu\|, \end{aligned}$$

i.e. $c(\nu; G^*) \leq 5$.

Finally, we would like to give another application of our Lemma. We recall that a maximal convolution operator G^* is said to be of weak type (q, q) , $q > 1$, if there exists a constant c such that

$$(6) \quad m\{G^* f > \lambda\} \leq (c\|f\|_q/\lambda)^q$$

for every $\lambda > 0$ and $f \in L^q (= L^q(\mathbf{R}^h, m))$, where $\|\cdot\|_q$ is the usual L^q norm. In [2, Theorem 2], Carrillo and de Guzman show that a sufficient condition for G^* to be of weak type (q, q) is

$$(7) \quad m\{G^* \nu > \lambda\} \leq c\|\nu\|/\lambda^q$$

for some $c \geq 0$, every $\lambda > 0$ and every $\nu = \sum_{i=1}^k \delta_{x_i}$, with $x_i \neq x_j$ if $i \neq j$. Also they give an example to show that condition (7) fails to be necessary.

In this connection we have

THEOREM 2. *Let G^* be the maximal convolution operator associated with $(g_i)_{i \in I}$, a countable collection of kernels in L^1 . Assume that G^* satisfies condition (7) for some $q > 1$. Then $g_i = 0$ for every $i \in I$.*

PROOF. Condition (7) is equivalent to

$$(8) \quad [c(\nu; G^*)]^q \leq c\|\nu\|^{1-q}$$

for every $\nu = \sum_{i=1}^k \delta_{x_i}$, with $x_i \neq x_j$ if $i \neq j$, where $c(\nu; G^*)$ is defined by (2). For $k \in \mathbf{N}$, take $\nu_k = \sum_{i=1}^k \delta_{x_{i,k}}$, where $x_{1,k}, \dots, x_{k,k}$ are k distinct points in the ball centered at the origin and with radius $1/k$. Then $\nu_k/k \xrightarrow{w} \delta_0$. Now, by (8) and (3), it follows that

$$c(\nu_k/k; G^*) = c(\nu_k; G^*) \leq ck^{1-q}$$

for every $k \in \mathbf{N}$. Hence, by the Lemma, $c(\delta_0; G^*) = 0$ and the theorem follows.

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