AVERAGE RADIAL LIMITS IN WEIGHTED HARDY SPACES

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ABSTRACT. Weighted Hardy spaces are defined in the unit disc by specifying the rate of growth of p th means near the boundary. Although a function in one of these spaces need have no radial limits, it is shown that in certain of these spaces "average radial limits" exist over an interval on the boundary. An integral representation in terms of these average radial limits is given, with an application to the question of existence of (pointwise) radial limits.

0. Introduction. Let H_α^p ($0 < p \leq \infty$) denote the space of functions f analytic in the unit disc $D \subseteq \mathbb{C}$ satisfying the growth condition

$$M_p(f, r) \leq c(1 - r)^{-\alpha} \quad (0 < r < 1),$$

where

$$M_p(f, r) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p} \quad \text{for } 0 < p < \infty,$$

$$M_\infty(f, r) = \sup_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.$$

Defining

$$\|f\|_{p,\alpha} = \sup_{0 < r < 1} (1 - r)^\alpha M_p(f, r)$$

gives a norm (if $p \geq 1$) on H_α^p making it into a Banach space (for completeness cf. [M]; for general background cf. [HL]).

For f analytic in D , $t > 0$, $0 < \rho < 1$ and $0 \leq \theta \leq 2\pi$, we set

$$A_t^\rho f(e^{i\theta}) = \frac{1}{2t} \int_{-t}^t f(\rho e^{i(\theta-s)}) ds,$$

and define

$$A_t f(e^{i\theta}) = \lim_{\rho \rightarrow 1^-} A_t^\rho f(e^{i\theta}),$$

provided the limit exists. If $A_t f(e^{i\theta})$ exists, we call this an "average radial limit" of f at the point $z = e^{i\theta}$.

The purpose of this note is to give in Theorem 1 and its corollary a precise description of those H_α^p spaces whose functions admit average radial limits at every boundary point. This is then applied to produce a theorem of Fatou type for these spaces.

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1. Existence of average radial limits. For $0 < \beta \leq 1$, let Λ_β denote the space of functions analytic in D , continuous in $\bar{D}$, and satisfying a Lipschitz condition of order β :

$$|f(e^{is}) - f(e^{it})| \leq c_\beta |s - t|^\beta.$$

Theorem 1 will be an immediate consequence of the following theorem of Hardy and Littlewood (Theorem 5.1 in [D]):

THEOREM A. For $0 < \beta \leq 1$ we have $f \in \Lambda_\beta$ if and only if $f' \in H_{1-\beta}^\infty$; further

$$c_\beta \|f\|_{\Lambda_\beta} \leq |f(0)| + \|f'\|_{\infty, 1-\beta} \leq c'_\beta \|f\|_{\Lambda_\beta}.$$

THEOREM 1. If $f \in H_\alpha^\infty$, $\alpha < 1$, then $A_t f(e^{i\theta})$ exists for $0 < t \leq \pi$, $0 \leq \theta \leq 2\pi$; also

$$|A_t f(e^{i\theta})| \leq c_\alpha (\|f\|_{\infty, \alpha} / t^\alpha).$$

PROOF. Let $f \in H_\alpha^\infty$. We may suppose $\theta = 0$ and $f(0) = 0$. Choose F analytic in D so that $F'(z) = f(z)/z$. Note that $f(z)/z \in H_\alpha^\infty$, so Theorem A implies $F \in \Lambda_{1-\alpha}$. Now

$$\begin{aligned} A_t^p f(1) &= \frac{1}{2\pi i} \int_{-t}^t \frac{f(\rho e^{i\theta})}{\rho e^{i\theta}} i \rho e^{i\theta} d\theta \\ &= \frac{1}{2\pi i} \int_{\rho e^{-it}}^{\rho e^{it}} \frac{f(z)}{z} dz = \frac{1}{2\pi i} (F(\rho e^{it}) - F(\rho e^{-it})), \end{aligned}$$

so $A_t f(1) = (1/2\pi i)(F(e^{it}) - F(e^{-it}))$ exists and

$$|A_t f(1)| \leq (1/2t) t^{1-\alpha} \|F\|_{\Lambda_{1-\alpha}} \leq c_\alpha (\|f\|_{\infty, \alpha} / t^\alpha).$$

This completes the proof.

COROLLARY. If $f \in H_\alpha^p$, $\alpha < 1 - 1/p$, then $A_t f(e^{i\theta})$ exists and

$$|A_t f(e^{i\theta})| \leq c_{\alpha, p} (\|f\|_{p, \alpha} / t^{\alpha+1/p}).$$

PROOF. Another theorem of Hardy and Littlewood (Theorem 5.9 in [D]) states that $H_\alpha^p \subseteq H_{\alpha+1/p}^\infty$; the corollary follows.

This corollary is sharp in the following sense: Let $f(z) = (1 - z)^{-1}$. Then $f \in H_\alpha^p$ whenever $\alpha \geq 1 - 1/p$ ($p > 1$) although $A_t f(e^{i\theta})$ fails to exist. (In §3 we will describe an $f \in H_1^\infty$ such that $A_t f(e^{i\theta})$ exists for no value of θ .)

2. A Fatou theorem. We now show how $f(r)$ ($0 \leq r < 1$) can be recovered from $\{A_t f(1): 0 < t \leq \pi\}$ by an "integrated Poisson formula":

Let $P_r(\theta) = (1 - r^2)/(1 - 2r \cos(\theta) + r^2)$ be the Poisson kernel and let $P'_r(\theta) = (\partial/\partial \theta)P_r(\theta)$. Note that $P'_r \leq 0$ on $[0, \pi]$. For $0 \leq r < 1$ define a measure μ_r on $[0, \pi]$ by

$$\int_0^\pi \phi(t) d\mu_r(t) = \frac{1-r}{1+r} \phi(\pi) - \frac{1}{\pi} \int_0^\pi t P'_r(t) \phi(t) dt.$$

The following lemmas are each proved by integration by parts.

LEMMA 1. The measures μ_r satisfy

- (a) $\mu_r \geq 0$,
- (b) $\mu_r([0, \pi]) = 1$,
- (c) for any $\delta > 0$, $\lim_{r \rightarrow 1} \mu_r([\delta, \pi]) = 0$.

LEMMA 2. Let $f \in L^1(T)$ and $A_t f(e^{i\theta}) = (1/2t) \int_{-t}^t f(e^{i(\theta-s)}) ds$. Then

$$P[f](re^{i\theta}) = \int_0^\pi A_t f(e^{i\theta}) d\mu_r(t) \quad (0 \leq r < 1).$$

(Here $P[f](re^{i\theta}) = (1/2\pi) \int_{-\pi}^\pi f(e^{i(\theta-s)}) P_r(s) ds$, as usual.)

THEOREM 2. If $f \in H_\alpha^\infty$, $\alpha < 1$, then

$$f(re^{i\theta}) = \int_0^\pi A_t f(e^{i\theta}) d\mu_r(t) \quad (re^{i\theta} \in D).$$

PROOF. Let $0 < p < 1$ and $f_p(z) = f(\rho z)$. Lemma 2 shows that

$$f_p(re^{i\theta}) = \int_0^\pi A_t f_p(e^{i\theta}) d\mu_r(t).$$

But $A_t f_p = A_t^\rho f \rightarrow A_t f$ as $\rho \rightarrow 1$, and for all ρ we have

$$|A_t f_p| \leq c_\alpha (\|f_p\|_{\infty, \alpha} / t^\alpha) \leq c_\alpha (\|f\|_{\infty, \alpha} / t^\alpha).$$

Since $\int_0^\pi t^{-\alpha} d\mu_r(t) < \infty$, the theorem follows by dominated convergence.

COROLLARY. Suppose $f \in H_\alpha^\infty$, $\alpha < 1$. If $\lim_{t \rightarrow 0^+} A_t f(e^{i\theta}) = 0$, then $\lim_{r \rightarrow 1^-} f(re^{i\theta}) = 0$.

PROOF. Lemma 1.

The converse of this corollary is false: For any $p < \infty$ there exists $f \in H^p = H_\rho^p \subseteq H_{1/\rho}^\infty$ such that $\lim_{r \rightarrow 1^-} f(r) = 0$, but $\lim_{t \rightarrow 0^+} A_t f(1)$ does not exist. If, on the other hand, $f \in H^\infty$, then a theorem of Loomis [L] states that $\lim_{r \rightarrow 1^-} f(r) = 0$ implies $\lim_{t \rightarrow 0^+} A_t f(1) = 0$. This last statement has recently been extended to f in the Bloch space; see [U].

3. An example. The example mentioned at the end of §1 leaves open the possibility that, e.g., $A_t f(e^{i\theta})$ should exist for almost all θ if $f \in H_1^\infty$. This is not the case:

EXAMPLE. Let $0 < t < \pi$. There exists $f \in H_1^\infty$ such that $A_t f(e^{i\theta})$ exists for no value of θ .

PROOF. Fix $t \in (0, \pi)$. Let $S = \{n = 1, 2, \dots : |\sin(nt)| \geq 1/2\}$. Note that S is infinite, since $0 < t < \pi$. For $n = 0, 1, \dots$ let $e_n(z) = z^n$; note that for $n \in S$,

$$(1) \quad |e_n(e^{it}z) - e_n(e^{-it}z)| \geq |z|^n.$$

Suppose $n_1, \dots, n_N \in S$ and $0 < r_1 < r'_1 < \dots < r_N < r'_N < 1$ have been chosen so that if $F_N = \sum_{j=1}^N e_{n_j}$ and $G_N(z) = F_N(e^{it}z) - F_N(e^{-it}z)$, then

$$(2) \quad (1 - |z|)|F'_N(z)| < 1 \quad (z \in D)$$

and

$$(3) \quad |G_N(r_j e^{i\theta}) - G_N(r'_j e^{i\theta})| > 1/2 \quad (j = 1, 2, \dots, N; 0 \leq \theta \leq 2\pi).$$

Now if $n_{N+1} \in S$ is large enough, (2) holds with $N + 1$ in place of N . It follows from (1) that if $n_{N+1} \in S$ is large enough, then we may choose r_{N+1} and r'_{N+1} , with $r'_N < r_{N+1} < r'_{N+1} < 1$, such that (3) holds with $N + 1$ in place of N .

Let $F = \lim_{N \rightarrow \infty} F_N$, $G = \lim_{N \rightarrow \infty} G_N$, so $G(z) = F(e^{it}z) - F(e^{-it}z)$. Let $f(z) = zF'(z)$. Then (2) implies that $f \in H_1^\infty$. On the other hand, the proof of Theorem 1 shows that

$$|A'_t f(e^{i\theta}) - A'_{t'} f(e^{i\theta})| = (1/2t) |G(r_j e^{i\theta}) - G(r'_j e^{i\theta})| \geq 1/4t,$$

so $\lim_{t \rightarrow 1} A'_t f(e^{i\theta})$ does not exist.

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