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SPECTRA OF SOME DOMAINS IN COMPACT LIE GROUPS AND THEIR APPLICATIONS[1]

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ABSTRACT. In this paper, we determine explicitly the spectra of the Dirichlet problems of some domains in simply connected compact simple Lie groups. As their applications, combining results of Hoffman [6] and Mori [10], we can state some stability conditions of these domains for the standard minimal isometric immersions into unit spheres.

1. Introduction and results. Let M be a simply connected compact simple Lie group and let T be its maximal torus. We give a bi-invariant Riemannian metric g on M from the Killing form B of the Lie algebra $\mathfrak{m}$ of M by

$$g_m(X_m, Y_m) = -B(X, Y), \qquad X, Y \in \mathfrak{m},\ m \in M,$$

where X_m, Y_m are tangent vectors of M at m corresponding to X, Y. Let $d(x, y)$ be the distance of (M, g) between two points x, y in M. Then it is known (cf. Crittenden [4] and Sakai [13]) that the cut locus C of the identity e in M satisfies

$$C = \bigcup_{x \in M} xC(T)x^{-1},$$

where $C(T)$ is the cut locus of e in the flat torus T induced from the Riemannian metric g. For a positive number ε with $0 < \varepsilon < d(e, C)$, consider a domain $\Omega(\varepsilon)$ containing the cut locus C in M defined by

$$\Omega(\varepsilon) = \bigcup_{x \in M} x\Omega(\varepsilon, T)x^{-1}, \qquad \Omega(\varepsilon, T) = \{t \in T; d(t, C(T)) < \varepsilon\}.$$

These domains $\Omega(\varepsilon)$, which are invariant under all the inner automorphisms of M, shrink to the cut locus C as $\varepsilon \to 0$.

Now let Δ be the Laplace-Beltrami operator of (M, g) acting on the space $C^\infty(M)$ of smooth functions on M, and for every ε with $0 < \varepsilon < d(e, C)$, let us consider the following Dirichlet problem for the above domains:

$$(\#)_\varepsilon \qquad \begin{cases} \Delta u + \lambda u = 0 & \text{on } M \setminus \overline{\Omega(\varepsilon)}, \\ u = 0 & \text{on } \Omega(\varepsilon). \end{cases}$$

For a solution u of the Dirichlet problem $(\#)_\varepsilon$, define a function u^0 on M by

$$u^0(x) = \int_M u(yxy^{-1})\,dy,$$

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where dy is the Haar measure on M normalized by $\int_M dy = 1$. Then, if u^0 does not vanish identically, u^0 is also a solution of $(\#)_\varepsilon$ which is a *zonal spherical* function of M, i.e., invariant under all the inner automorphisms of M.

In this paper, we determine the spectra of the Dirichlet problem $(\#)_\varepsilon$, which have zonal spherical eigenfunctions as follows

THEOREM 1. *Let M be a simply connected compact simple Lie group and let Δ be the Laplace-Beltrami operator of the Riemannian metric g of M induced from the negative of the Killing form B of the Lie algebra $\mathfrak{m}$ of M. Then for every ε with $0 < \varepsilon < d(e, C)$, the eigenvalues of the Dirichlet problem $(\#)_\varepsilon$ which have zonal spherical eigenfunctions are given by*

$$(1) \qquad \left\{\frac{d(e,C)}{d(e,C)-\varepsilon}\right\}^2 |\Lambda+\delta|^2 - |\delta|^2, \qquad \Lambda \in \mathbf{D},$$

and the corresponding zonal spherical eigenfunctions $u_{\Lambda,\varepsilon}$ are described explicitly by

(2)

$$u_{\Lambda,\varepsilon}(\exp H) = \begin{cases} \xi_{\Lambda+\delta}\left(\exp\left(\dfrac{d(e,C)}{d(e,C)-\varepsilon}H\right)\right)/\xi_\delta(\exp H), & \exp H \in T\setminus\Omega(\varepsilon,T), \\ 0, & \exp H \in \Omega(\varepsilon,T). \end{cases}$$

Here $\mathbf{D}$ is the set of all dominant integral forms on the Lie algebra $\mathfrak{t}$ of T, δ is half the sum of all positive roots, $|\cdot|$ is the inner product of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ induced from the negative of the Killing form, and ξ_λ, $\lambda \in \mathbf{D}$, are the alternating characters of T (cf. §2).

Theorem 1 implies immediately

COROLLARY 1. *Under the assumptions of Theorem 1, the first eigenvalue $\lambda_1(\varepsilon)$ of the Dirichlet problem $(\#)_\varepsilon$, $0 < \varepsilon < d(e, C)$, is given by*

$$\left\{\frac{d(e,C)}{d(e,C)-\varepsilon}\right\}^2 |\delta|^2 - |\delta|^2 = \left\{\frac{d(e,C)}{d(e,C)-\varepsilon}\right\}^2 \frac{d}{24} - \frac{d}{24},$$

where $d = \dim M$ (cf. [15, p. 291]). The corresponding eigenfunction with the eigenvalue $\lambda_1(\varepsilon)$ is $u_{0,\varepsilon}$.

REMARK. In the case $S^3 = \mathrm{SU}(2)$, the same formula as Theorem 1 was obtained in [3, p. 201]. Chavel and Feldman [3] also investigated the behavior of the eigenvalues $\lambda_i(\varepsilon)$ of the Dirichlet problems of the domains $X\setminus\overline{\Omega(\varepsilon)}$, where $\Omega(\varepsilon) = \{x \in X; d(x, Y) < \varepsilon\}$ for every compact Riemannian manifold X and a closed submanifold Y of X with codim $\geqslant 2$. A more precise behavior of the first eigenvalue was obtained in Ozawa [12] and Matsuzawa and Tanno [9].

As a geometric application of Corollary 1, we can state some stability conditions of those domains $M\setminus\overline{\Omega(\varepsilon)}$ in M for the standard minimal isometric immersions x_k of M into the unit sphere as follows.

Let $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots\}$ be the set of all mutually distinct eigenvalues of the negative of the Laplace-Beltrami operator Δ acting on $C^\infty(M)$.

Let V^k, $k = 1, 2, \ldots$, be the eigenspace with the eigenvalue λ_k, and put $m(k) + 1 = \dim V^k$. We choose an orthonormal basis $\{f_j\}_{j=0}^{m(k)}$ of V^k consisting of real valued functions with respect to the inner product $(\varphi, \psi) = \int_M \varphi(x)\psi(x)\,d\mu(x)$, where $d\mu(x)$ is the Haar measure of M normalized by $\int_M d\mu(x) = m(k) + 1$. Consider the mapping x_k of M into the Euclidean space $\mathbf{R}^{m(k)+1}$ defined by

$$x_k(p) = \left(f_0(p), f_1(p), \ldots, f_{m(k)}(p)\right), \qquad p \in M.$$

Then it turns out that the image of x_k is contained in the unit sphere $S^{m(k)}$, moreover the mapping x_k is a minimal isometric immersion of $(M, \lambda_k g/d)$, $d = \dim M$, into the unit sphere $S^{m(k)}$ with the standard Riemannian metric of constant curvature 1 (cf. [**8**]) since M is a simple Lie group.

For a piecewise smooth domain D in M, we call D *stable* for the minimal immersion x_k if, for all normal variations D_t which fix the boundary ∂D, the function $V(t) = \text{Volume}\, D_t$ satisfies $V''(0) > 0$. Combining Corollary 1 with results of Hoffman [**6**] and Mori [**10**] we have

COROLLARY 2. *Under the situations of Theorem* 1, *if a positive number* ε *satisfies*

$$d(e, C) > \varepsilon > d(e, C) - d(e, C)\left\{\frac{24\lambda_k}{d}\left(\|A\|^2 + d\right) + 1\right\}^{-1/2},$$

then D is stable for the minimal isometric immersion x_k *for every* $D \subset M \setminus \overline{\Omega(\varepsilon)}$. *Here* $\|A\|^2$ *is the square of the length of the second fundamental form of the immersion* x_k.

REMARK. In the case of $M = \text{Sp}(n)$ and $k = 1$, it is then known (cf. Nagura [**11**], Kobayashi and Takeuchi [**7**]) that $d = n(2n + 1)$,

$$\|A\|^2 = n(n-1)(n+1)(2n+1), \quad \text{and} \quad \lambda_1 = (2n+1)/(4n+4).$$

Therefore for every $D \subset M \setminus \overline{\Omega(\varepsilon)}$, D is stable for the immersion x_1 if $d(e, C) > \varepsilon > d(e, C)\{1 - \sqrt{(n+1)/(7n+1)}\}$, in particular, if $d(e, C) > \varepsilon > 0.623 d(e, C)$.

2. Preliminaries. Since we will use the precise formula of the radial part (cf. [**2**]) of the Laplace-Beltrami operator and the structure of the cut loci C and $C(T)$ (cf. [**13**]) in the proof of Theorem 1, we have to prepare some notation.

2.1. Let M be a simply connected compact simple Lie group, and let T be a maximal torus in M. Let $\mathfrak{m}$ (resp. $\mathfrak{t}$) be the Lie algebra of M (resp. T). Since the Killing form B is negative definite on $\mathfrak{m}$, we define an $\text{Ad}(M)$-invariant positive definite inner product $(\ ,\ )$ on $\mathfrak{m}$ by $(X, Y) = -B(X, Y)$, $X, Y \in \mathfrak{m}$, which induces a bi-invariant Riemannian metric g on M as in the introduction. Let Σ be the root system of the complexification $\mathfrak{m}^{\mathbf{C}}$ of $\mathfrak{m}$ with respect to $\mathfrak{t}$, i.e., the set of nonzero elements α of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ such that $\{E \in \mathfrak{m}^{\mathbf{C}}; [H, E] = \sqrt{-1}\,\alpha(H)E$ for all $H \in \mathfrak{t}\}$ is not zero. We give a lexicographic order $>$ on Σ and let Σ_+ be the set of all positive roots. Let α^0 be the highest root of Σ_+ with respect to the order $>$. Put $\overline{\mathfrak{t}^+} = \{H \in \mathfrak{t}; \alpha(H) \geqslant 0$ for all $\alpha \in \Sigma_+\}$. Then the cut locus C of the identity e in (M, g) is given (cf. Sakai [**13**]) by

(2.1) $$C = \bigcup_{x \in M} x C(T) x^{-1}.$$

Here $C(T)$ is the cut locus of the flat torus T induced from the Riemannian metric g which is given (cf. Takeuchi [14] and Sakai [13]) by

(2.2) $$C(T) = \exp \tilde{C}(\mathfrak{t}),$$

(2.3) $$\tilde{C}(\mathfrak{t}) = \bigcup_{s \in W} s\{H \in \overline{\mathfrak{t}^+}; \alpha^0(H) = 2\pi\},$$

where W is the Weyl group of M.

Put

(2.4) $$\tilde{D}^+(\mathfrak{t}) = \{H \in \overline{\mathfrak{t}^+}; \alpha^0(H) \leqslant 2\pi\}, \quad \tilde{D}(\mathfrak{t}) = \bigcup_{s \in W} s\tilde{D}^+(\mathfrak{t}),$$

and $\tilde{D} = \bigcup_{x \in M} \mathrm{Ad}(x)\tilde{D}(t)$. Then $\tilde{C}(\mathfrak{t})$ is the boundary $\partial\tilde{D}(\mathfrak{t})$ of $\tilde{D}(\mathfrak{t})$, both the exponential mappings exp: $\tilde{D}(\mathfrak{t}) \to T$ and exp: $\tilde{D} \to M$ are onto mappings, and the restriction to the interior of $\tilde{D}$ is a diffeomorphism. Moreover the distance $d(e,C)$ between the identity e and the cut locus C is given by

(2.5) $$d(e,C) = 2\pi/|\alpha^0|.$$

Here $|\cdot|$ is the norm of the inner product $(\ ,\ )$ on $\mathfrak{t}^*$ induced from the inner product $(\ ,\ )$ on $\mathfrak{t}$ by $(\lambda, \mu) = (H_\lambda, H_\mu)$, λ, $\mu \in \mathfrak{t}^*$, where $H_\lambda \in \mathfrak{t}$, $\lambda \in \mathfrak{t}^*$, is the unique element in $\mathfrak{t}$ satisfying $(H_\lambda, H) = \lambda(H)$ for every $H \in \mathfrak{t}$. Note that the distance $d(x, y)$, x, $y \in T$, coincides with the one with respect to the Riemannian metric on T induced from the metric g on M (see Remark in [5, p. 80]). In fact, since T is totally geodesic in M, we have only to show the existence of a distance minimizing geodesic in T joining e and every x in T, but it follows immediately from Theorem 7.9(ii) and Lemma 7.10 in [5].

Then we have

LEMMA 2.1. *For every ε with $0 < \varepsilon < d(e,C) = 2\pi/|\alpha^0|$,*
(i) *the set $\Omega(\varepsilon, T) = \{t \in T; d(t, C(T)) < \varepsilon\}$ is given by*

$$\Omega(\varepsilon, T) = \exp \tilde{\Omega}(\varepsilon, \mathfrak{t}),$$

$$\tilde{\Omega}(\varepsilon, \mathfrak{t}) = \bigcup_{s \in W} s\left\{H \in \overline{\mathfrak{t}^+}; 2\pi\left(1 - \varepsilon\frac{|\alpha^0|}{2\pi}\right) < \alpha^0(H) \leqslant 2\pi\right\}.$$

(ii) *The set $M \setminus \overline{\Omega(\varepsilon)}$ is given by*

$$M \setminus \overline{\Omega(\varepsilon)} = \bigcup_{x \in M} x \exp \tilde{D}^+(\varepsilon) x^{-1},$$

where $\tilde{D}^+(\varepsilon) = \{H \in \overline{\mathfrak{t}^+}; \alpha^0(H) < 2\pi(1 - \varepsilon|\alpha^0|/2\pi)\}$.

(iii) *$d(e,C)/(d(e,C) - \varepsilon) \cdot \tilde{D}^+(\varepsilon) = \{H \in \overline{\mathfrak{t}^+}; \alpha^0(H) < 2\pi\}$. Here, for every $r > 0$, $r \cdot \tilde{D}^+(\varepsilon)$ means the set $\{rH; H \in \tilde{D}^+(\varepsilon)\}$.*

PROOF. (i) By the definition of $\tilde{\Omega}(\varepsilon, \mathfrak{t})$, (2.3) and the invariance of the distance d under the inner automorphisms of M, we have

$$\tilde{\Omega}(\varepsilon, \mathfrak{t}) = \bigcup_{s \in W} s\{H \in \tilde{D}^+(\mathfrak{t}); d(\exp H, \exp \tilde{C}(t)) < \varepsilon\}.$$

We denote $d_e(X, Y) = |X - Y|$ for $X, Y \in \mathfrak{t}$. Then, for each $H \in \tilde{D}^+(\mathfrak{t})$,

(2.6) $$\begin{aligned} d(\exp H, \exp \tilde{C}(\mathfrak{t})) &= d(\exp H, \exp(\tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+})) \\ &= d_e(H, \tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+}). \end{aligned}$$

In fact, putting $\Gamma = \{X \in \mathfrak{t}; \exp X = e\}$, we have

$$d(\exp H, \exp \tilde{C}(\mathfrak{t})) = d_e(H, \tilde{C}(\mathfrak{t}) + \Gamma)$$

and

$$d\left(\exp H, \exp\left(\tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+}\right)\right) = d_e\left(H, \left(\tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+}\right) + \Gamma\right).$$

Since $(\tilde{C}(\mathfrak{t}) + \Gamma) \cap \tilde{D}(\mathfrak{t}) = \tilde{C}(\mathfrak{t})$ and $((\tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+}) + \Gamma) \cap \tilde{D}(\mathfrak{t}) = \tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+}$,

$$d_e(H, \tilde{C}(\mathfrak{t}) + \Gamma) = d_e(H, \tilde{C}(\mathfrak{t}))$$

and

$$d_e\left(H, \left(C(\mathfrak{t}) \cap \overline{\mathfrak{t}^+}\right) + \Gamma\right) = d_e\left(H, \tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+}\right).$$

Since $H \in \tilde{D}^+(\mathfrak{t})$, we have $d_e(H, \tilde{C}(\mathfrak{t})) = d_e(H, \tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+})$. For the proof of the second equality, choose an element X in $\tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+}$ such that

$$d\left(\exp H, \exp\left(\tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+}\right)\right) = d(\exp H, \exp X).$$

Then $d(\exp H, \exp X) = d(e, \exp(-H + X))$ and $-H + X \in \tilde{D}(\mathfrak{t})$, because $0 \leqslant \alpha(H), \alpha(X) \leqslant 2\pi$ for $\alpha \in \Sigma_+$, and the definition (2.4) of $\tilde{D}(t)$. Then

$$d(e, \exp(-H + X)) = |-H + X|,$$

which implies $d(\exp H, \exp(\tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+})) \geqslant d_e(H, \tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+})$. The converse inequality is clear.

By (2.3) and (2.6), we have

$$\begin{aligned} \{H \in \tilde{D}^+(\mathfrak{t}); d(\exp H, \exp \tilde{C}(\mathfrak{t})) < \varepsilon\} &= \left\{H \in \tilde{D}^+(\mathfrak{t}); d_e\left(H, \tilde{C}(\mathfrak{t}) \cap \overline{\mathfrak{t}^+}\right) < \varepsilon\right\} \\ &= \left\{(1 - r)\frac{2\pi H_{\alpha^0}}{(\alpha^0, \alpha^0)} + X; |r| < \frac{\varepsilon|\alpha^0|}{2\pi}, X \in \mathfrak{t}, \alpha^0(X) = 0\right\} \cap \tilde{D}^+(t) \\ &= \left\{H \in \overline{\mathfrak{t}^+}; 2\pi\left(1 - \varepsilon\frac{|\alpha^0|}{2\pi}\right) < \alpha^0(H) \leqslant 2\pi\right\}. \end{aligned}$$

For (ii), we have only to show $X = Y$ when $g_1 \exp X g_1^{-1} = g_2 \exp Y g_2^{-1}$, $X \in \tilde{D}^+(\varepsilon)$, $Y \in \tilde{D}^+(\mathfrak{t})$, $g_1, g_2 \in M$. But in this case, we have $\exp X = \exp sY$ for some $s \in W$ by Lemma 7.10 in [5]. Since $sY \in \tilde{D}(\mathfrak{t})$ and $X \in \tilde{D}^+(\varepsilon)$, $\exp X = \exp sY$ implies $X = sY$, and then $X = Y$. (iii) follows immediately from (ii). Q.E.D.

2.2. For $\lambda \in \mathfrak{t}^*$, $\lambda \neq 0$, put $H_\lambda^* = 2H_\lambda/(\lambda, \lambda)$. Then since M is simply connected, the lattice $\Gamma = \{H \in \mathfrak{t}; \exp H = e\}$ is given by $\Gamma = 2\pi\sum_{i=1}^{l}\mathbf{Z}H_{\alpha_i}^*$, where $\{\alpha_i\}_{i=1}^{l}$ is a fundamental system of Σ with respect to the order $>$, and $l = \dim T$. Put

$$\begin{aligned} I &= \left\{\lambda \in \mathfrak{t}^*; \lambda\left(H_{\alpha_i}^*\right) \in \mathbf{Z}, i = 1, \ldots, l\right\} \\ &= \left\{\lambda \in \mathfrak{t}^*; \lambda(\Gamma) \subset 2\pi\mathbf{Z}\right\}, \\ \mathbf{D} &= \left\{\lambda \in I; (\lambda, \alpha) \geqslant 0 \text{ for every } \alpha \in \Sigma^+\right\}. \end{aligned}$$

An element of $\mathbf{D}$ is called a *dominant integral* form on $\mathfrak{t}$. For $\lambda \in I$, define a function ξ_λ on T, called the *alternating character*, by

$$\xi_\lambda(\exp H) = \sum_{s \in W} (-1)^s e^{s\lambda(H)}, \qquad H \in \mathfrak{t}.$$

Put $\delta = \frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$. Then δ belongs to **D**. Moreover it is known that

$$\xi_\delta(\exp H) = \prod_{\alpha\in\Sigma^+}\left(e^{\sqrt{-1}\,\alpha(H)/2} - e^{-\sqrt{-1}\,\alpha(H)/2}\right),$$

every ξ_λ, $\lambda \in I$, can be divided by ξ_δ, and $\xi_{\Lambda+\delta}/\xi_\delta$, $\lambda \in \mathbf{D}$, coincides with the restriction to T of the character χ_Λ of the irreducible unitary representation of M with highest weight Λ (cf. [**14**]). For every C^∞ zonal spherical function f on M, let $\bar{f}$ be its restriction to T. Then $\bar{f}(\exp sH) = \bar{f}(\exp H)$, $s \in W$, $H \in \mathfrak{t}$, and we have (cf. Berezin [**2**] or [**14**])

$$\xi_\delta(\overline{\Delta f}) = \left\{\Delta_0 + |\delta|^2\right\}(\xi_\delta \bar{f}) \tag{2.7}$$

on T, where Δ_0 is the standard Laplacian on T induced from the Euclidean Laplacian of $\mathfrak{t}$ with respect to the inner product (,).

3. **Proof of Theorem 1.** For $0 < \varepsilon < d(e, C)$, assume that a zonal spherical function u on M satisfies

$$\begin{cases} \Delta u + \lambda u = 0 & \text{on } M \setminus \overline{\Omega(\varepsilon)}, \\ u = 0 & \text{on } \Omega(\varepsilon). \end{cases} \tag{3.1}$$

Then by (2.7) we have

$$\begin{cases} \left(\Delta_0 + |\delta|^2\right)(\xi_\delta \bar{u}) + \lambda \xi_\delta \bar{u} = 0 & \text{on } T \setminus \overline{\Omega(\varepsilon, T)}, \\ \bar{u} = 0 & \text{on } \Omega(\varepsilon, T). \end{cases}$$

Now define a function $(\xi_\delta \bar{u})_\varepsilon$ on T by

$$(\xi_\delta \bar{u})_\varepsilon(\exp H) = (\xi_\delta \bar{u})\left(\exp\left(\frac{d(e,C) - \varepsilon}{d(e,C)} H\right)\right), \qquad H \in \tilde{D}(\mathfrak{t}).$$

It is well defined on T because of Lemma 3.1(iii), and $\bar{u} = 0$ on $\Omega(\varepsilon, T)$. Also define a function $\widetilde{(\xi_\delta \bar{u})_\varepsilon}$ on $\tilde{D}(\mathfrak{t})$ by

$$\widetilde{(\xi_\delta \bar{u})_\varepsilon}(H) = (\xi_\delta \bar{u})_\varepsilon(\exp H), \qquad H \in \tilde{D}(\mathfrak{t}).$$

Then the function $\widetilde{(\xi_\delta \bar{u})_\varepsilon}$ satisfies

$$\begin{cases} \Delta_0 \widetilde{(\xi_\delta \bar{u})_\varepsilon} + \left\{\dfrac{d(e,C) - \varepsilon}{d(e,C)}\right\}^2 \left(|\delta|^2 + \lambda\right)\widetilde{(\xi_\delta \bar{u})_\varepsilon} = 0, \\ \qquad\qquad \text{on the interior of } \tilde{D}(t), \\ \widetilde{(\xi_\delta \bar{u})_\varepsilon} = 0 \quad \text{on } \partial \tilde{D}(\mathfrak{t}). \end{cases}$$

Moreover, $\widetilde{(\xi_\delta \bar{u})_\varepsilon} = 0$ on $\tilde{D}^+(\mathfrak{t})$ since $\xi_\delta = 0$ on $\partial \tilde{D}^+(\mathfrak{t})$. Therefore $\widetilde{(\xi_\delta \bar{u})_\varepsilon}$ is the eigenfunction of the Dirichlet problem for the domain $\tilde{D}^+(\mathfrak{t})$. Since the domain $\tilde{D}^+(\mathfrak{t})$ is a fundamental domain of the affine Weyl group of the Lie group M acting on $\mathfrak{t}$, by a theorem of Bérard [**1**], we have

$$\widetilde{(\xi_\delta \bar{u})_\varepsilon}(H) = \sum_{s\in W} (-1)^s e^{\sqrt{-1}\, s(\Lambda+\delta)(H)}$$

for some $\Lambda \in \mathbf{D}$, and $\{(d(e,C) - \varepsilon)/d(e,C)\}^2(|\delta|^2 + \lambda) = |\Lambda + \delta|^2$. Therefore we obtain

$$\lambda = \left\{ \frac{d(e,C)}{d(e,C) - \varepsilon} \right\}^2 |\Lambda + \delta|^2 - |\delta|^2 \tag{3.2}$$

and

$$u(\exp H) = \begin{cases} \xi_{\Lambda+\delta}\left(\exp\left(\dfrac{d(e,C)}{d(e,C) - \varepsilon} H\right)\right) / \xi_\delta(\exp H), & H \in \Omega(\varepsilon, \mathrm{t}), \\ 0, & H \notin \Omega(\varepsilon, \mathrm{t}). \end{cases} \tag{3.3}$$

Conversely, the function u defined by (3.3) is a zonal spherical function on M and satisfies (3.1) with the eigenvalue (3.2). We have proved Theorem 1.

References

1. P. H. Bérard, *Spectres et groupes cristallographiques* I: *Domaines euclidiens*, Invent. Math. **58** (1980), 179–199.
2. F. A. Berezin, *Laplace operators on semi-simple Lie groups*, Trudy Moskov. Mat. Obshch. **6** (1957), 371–463.
3. I. Chavel and E. A. Feldman, *Spectra of domains in compact manifolds*, J. Funct. Anal. **30** (1978), 198–222.
4. R. Crittenden, *Minimum and conjugate points in symmetric spaces*, Canad. J. Math. **14** (1962), 320–328.
5. S. Helgason, *Differentiai geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978.
6. D. Hoffman, *Lower bounds on the first eigenvalue of the Laplacian of Riemannian submanifolds*, Minimal Submanifolds and Geodesic, Kaigai Publ., Tokyo, 1978, pp. 61–73.
7. S. Kobayashi and M. Takeuchi, *Minimal imbeddings of R-spaces*, J. Differential Geom. **2** (1968), 203–215.
8. K. Mashimo, *Degree of the standard isometric minimal immersion of complex projective spaces into spheres*, Tsukuba J. Math. **4** (1980), 133–145.
9. T. Matsuzawa and S. Tanno, *Estimates of the first eigenvalue of a big cup domain of a* 2-*sphere*, Compositio Math. **47** (1982), 95–100.
10. H. Mori, *Notes on the stability of minimal submanifolds of Riemannian manifolds*, Yokohama Math. J. **25** (1977), 9–15.
11. T. Nagura, *On the lengths of the second fundamental forms of R-spaces*, Osaka J. Math. **14** (1977), 207–223.
12. S. Ozawa, *The first eigenvalue of the Laplacian on two dimensional Riemannian manifolds*, Tôhoku Math. J. **34** (1982), 7–14.
13. T. Sakai, *On the structure of cut loci in compact Riemannian symmetric spaces*, Math. Ann. (2) **235** (1978), 129–148.
14. M. Takeuchi, *Modern theory of spherical functions*, Iwanami, Tokyo, 1975 (in Japanese).
15. H. Urakawa, *The Heat equations on compact Lie groups*, Osaka J. Math. **12** (1975), 285–297.

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