ENGULFING AND FINITELY GENERATED GROUPS

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ABSTRACT. Let M be a simply connected 3-manifold and K a piecewise-linear, simple loop in the interior of M . It is shown that there is a piecewise-linear, homotopy 3-ball $\mathcal{B} \subset M$, such that $K \subset \mathcal{B}$ if and only if $\pi_1(M \setminus K)$ is finitely generated.

Let M be a 3-manifold and $K \subset M$. Say that K may be *engulfed* if there is a piecewise-linear, homotopy 3-ball $\mathcal{B} \subset M$ such that $K \subset \mathcal{B}$. The concept of engulfing is used in [6 and 2] to distinguish certain open, contractible 3-manifolds from $\mathbb{R}^3$. More specifically, each of the 3-manifolds contains a piecewise-linear, simple loop K which cannot be engulfed. This simple loop K also has the strange property that the fundamental group of the complement of K is not finitely generated. Below it is proved that this is not a coincidence. The proof is generalized from [5].

Interesting examples of M satisfying the hypotheses of the next theorem are contractible 3-manifolds, connected sums of contractible 3-manifolds, and the complements of Cantor sets in S^3 which have simply connected complements.

All manifolds and maps are assumed to be piecewise-linear.

THEOREM. *Let M be a simply connected, 3-manifold and K a piecewise-linear, simple loop in M . Then K may be engulfed if and only if $\pi_1(M \setminus K)$ is finitely generated.*

PROOF. First suppose that K may be engulfed by $\mathcal{B}$. Since K is piecewise-linear, clearly $\pi_1(\mathcal{B} \setminus K)$ is finitely generated. By Van Kampen's Theorem, $\pi_1(M \setminus K) \approx \pi_1(\mathcal{B} \setminus K)$, and one direction is proved.

Now suppose $\pi_1(M \setminus K)$ is finitely generated. By [3] there is a compact, piecewise-linear 3-manifold $Q \subset M \setminus K$, such that $\pi_1(Q) \rightarrow \pi_1(M \setminus K)$ is an isomorphism. Since M is orientable, Q too is orientable. Consider ∂Q . Since $\pi_1(Q) \neq 1$, Van Kampen's Theorem implies that ∂Q has at least one non-2-sphere component. Conversely, because Q is compact and orientable,

$$\text{rank}(\text{Image}(H_1(\partial Q) \rightarrow H_1(Q))) = \frac{1}{2} \text{rank}(H_1(\partial Q)).$$

Since $H_1(Q) \approx H_1(M \setminus K) \approx \mathbb{Z}$, $Q = T \amalg (\amalg S^2)$, where $T \approx S^1 \times S^1$.

Suppose $\pi_1(T) \rightarrow \pi_1(Q)$ is a monomorphism. Let $N(K)$ be a regular neighborhood of K in $M \setminus Q$. It will be shown that T is parallel to $\partial N(K)$. Each component of ∂Q separates M . Hence, each component of $M \setminus N(K) \setminus Q$ intersects Q in exactly one component of ∂Q . Let P be the component of $M \setminus N(K) \setminus Q$ containing $\partial N(K)$.

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Clearly, P is not simply connected. Since $\pi_1(Q) \rightarrow \pi_1(M \setminus \overset{\circ}{N}(K))$ is an isomorphism, Van Kampen's Theorem implies that $P \cap Q = T$. And since $\pi_1(T) \rightarrow \pi_1(Q)$ is a monomorphism, $\pi_1(T) \rightarrow \pi_1(P)$ is a monomorphism. Again by Van Kampen's Theorem $\pi_1(Q) \approx \pi_1(M \setminus K) \approx \pi_1(Q) *_{\pi_1(T)} \pi_1(P)$ and $\pi_1(T) \rightarrow \pi_1(P)$ is an isomorphism. By the lemma below, $\pi_1(\partial N(K)) \rightarrow \pi_1(P)$ is also an isomorphism.

Let μ, λ be the meridional and longitudinal curves on $\partial N(K)$; suppose μ and λ intersect transversely at one point. Similarly, let μ', λ' be corresponding curves on T which are freely homotopic to μ, λ respectively. By [4] there are properly embedded annuli $A_1, A_2 \subset P$ with $\partial A_1 = \mu \cup \mu'$ and $\partial A_2 = \lambda \cup \lambda'$. By cut and paste techniques suppose $A_1 \cap A_2$ is a single arc. Let $N(A_1 \cup A_2)$ be a regular neighborhood of $A_1 \cup A_2$ in P . Clearly, $\mathcal{B}' = Q \cup N(A_1 \cup A_2) \cup N(K)$ has boundary a disjoint union of S^2 's and $\pi_1(\mathcal{B}') = 1$. By drilling out holes in $\mathcal{B}'$ one may reduce the number of boundary components and produce the desired $\mathcal{B}$.

Finally, if $\pi_1(T) \rightarrow \pi_1(Q)$ is not a monomorphism, then an application of the Loop Theorem shows that

$$\mathbf{Z} \approx \pi_1(Q) \approx \pi_1(M \setminus K).$$

By Dehn's Lemma K bounds a disk in M , and a regular neighborhood of this disk is the desired $\mathcal{B}$. $\square$

That $\mathcal{B}$ may be chosen to be a (real) 3-ball is equivalent to the Poincaré Conjecture (see [1]).

LEMMA. *Let P be a connected 3-manifold with boundary (not necessarily compact). Let F and G be distinct components of ∂P such that $F \approx G \approx S^1 \times S^1$. If $\pi_1(F) \rightarrow \pi_1(P)$ is an isomorphism, then $\pi_1(G) \rightarrow \pi_1(P)$ is an isomorphism.*

PROOF. First suppose $\pi_1(G) \rightarrow \pi_1(P)$ is not a monomorphism. By the Loop Theorem there is an essential simple loop $C \subset G$ which bounds a disk D in P . Because $G \approx S^1 \times S^1$, there is a loop C' in G which intersects C transversely in one point. Hence each loop in P which is homotopic to C' must intersect D . Since each loop in F is disjoint with D , C' cannot be in the image of $\pi_1(F)$. But this is a contradiction. So $\pi_1(G) \rightarrow \pi_1(P)$ is a monomorphism.

Consequently, $[\pi_1(F) : \pi_1(G)]$ is finite. Let $p: \tilde{P} \rightarrow P$ be the finite-fold covering corresponding to $\pi_1(G)$. Let $\tilde{G}$ be a component of $p^{-1}(G)$. Then $\pi_1(\tilde{G}) \rightarrow \pi_1(\tilde{P})$ is an isomorphism by construction. In particular, $H_1(\tilde{G}) \rightarrow H_1(\tilde{P})$ is an isomorphism. Because $\tilde{P}$ is compact and orientable,

$$\text{rank}(\text{Image}(H_1(\partial \tilde{P}) \rightarrow H_1(\tilde{P}))) = \frac{1}{2} \text{rank}(H_1(\partial \tilde{P})).$$

Since the components of $p^{-1}(F \cup G)$ are closed, one sees that $p^{-1}(F \cup G)$ must have only two components. So $\tilde{P} = P$ and $\pi_1(G) \rightarrow \pi_1(P)$ is an isomorphism. $\square$

REFERENCES

1. R. H. Bing, *Necessary and sufficient conditions that a 3-manifold be S^3* , Ann. of Math. **68** (1958), 17–37.
2. Ralph H. Fox and Emil Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. **49** (1948), 979–990.
3. G. P. Scott, *Compact submanifolds of 3-manifolds*, J. London Math. Soc. (2) **7** (1973), 246–250.

4. A. Shapiro and J. H. C. Whitehead, *A proof and extension of Dehn's Lemma*, Bull. Amer. Math. Soc. **64** (1958), 174–178.
5. R. Skora, *Cantor sets in S^3 with simply connected complements*, Topology Appl. (1986) (to appear).
6. J. H. C. Whitehead, *A certain open manifold whose group is unity*, Quart. J. Math. (Oxford) **6** (1935), 268–279.

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