RINGS WITH A BOUNDED NUMBER OF GENERATORS FOR RIGHT IDEALS

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ABSTRACT. Let the ring S be a finitely generated module over a subring R of its center. Then it will be shown that S has the property that every right ideal can be generated by a bounded number of elements if and only if R has the property that every ideal can be generated by a bounded number of elements. As a corollary we show that a two-sided Noetherian affine ring satisfying a polynomial identity has the property that every right ideal can be generated by a bounded number of elements if and only if every left ideal can be generated by a bounded number of elements.

1. Introduction. Several authors have studied commutative Noetherian rings with the property that there exists an integer k such that every ideal can be generated by k or fewer elements. I. S. Cohen [3] showed that the classical Krull dimension of such rings is at most one, and that if R is a local Noetherian domain of Krull dimension one then there exists a positive integer k such that every ideal can be generated by k or fewer elements. In the noncommutative case J. T. Stafford [11] has shown that if R is a right and left Noetherian ring for which there exists a positive integer k such that every right ideal of R can be generated by k or fewer elements then R has classical Krull dimension at most one. An example of A. V. Jategaonkar [5] shows that the assumption that the ring is merely right Noetherian is not sufficient for Stafford's theorem.

J. D. Sally [9] calls a commutative ring bounded if there is a bound for the number of generators of all ideals. Using multiplicity theory she shows that if $R \subseteq S$ are commutative rings and if S is a finitely generated R -module, then R is bounded if and only if S is bounded. In this paper we will show that this theorem can be extended to the case where S is noncommutative and R is central in S . Our proof provides a new approach to the commutative case since we avoid the use of multiplicity theory. As an application we apply our theorem to affine rings satisfying a polynomial identity. Finally, we discuss several examples which indicate obstructions to some natural generalizations of our results.

2. Preliminaries. All rings will be assumed to have a multiplicative identity element, and all subrings of a ring will have the same identity. $Z(R)$ will denote the center of the ring R .

Let R be a ring and M a right R -module. If every right (resp. left) ideal of R can be generated by k or fewer elements we say that R satisfies r -Bnd(k) (resp. l -Bnd(k)). In the commutative case we omit the r or l . We say that R satisfies r -Bnd (resp. l -Bnd) if R satisfies r -Bnd(k) (resp. l -Bnd(k)) for some k . If every

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submodule of M can be generated by k or fewer elements we say that M satisfies $r\text{-Bnd}(k)$,

If every chain of prime ideals has at most two distinct elements, we say that the classical Krull dimension of R is at most one. Since we will not need to mention other Krull dimensions we omit the adjective classical from now on.

LEMMA 1. *Let $N \subseteq M$ be right R -modules.*

(i) *If M satisfies $r\text{-Bnd}(k)$ then N and M/N satisfy $r\text{-Bnd}(k)$.*

(ii) *If N satisfies $r\text{-Bnd}(k_1)$ and M/N satisfies $r\text{-Bnd}(k_2)$, then M satisfies $r\text{-Bnd}(k_1 + k_2)$.*

(iii) *If $M = m_1R + m_2R + \cdots + m_tR$ is a finitely generated right R -module and R satisfies $r\text{-Bnd}(k)$, then M satisfies $r\text{-Bnd}(k \cdot t)$.*

PROOF. (i) and (ii) are easy, and (iii) follows from (i) and (ii) by induction. $\square$

LEMMA 2. (i) *If the ring R_i satisfies $r\text{-Bnd}(k_i)$ for $i = 1, 2, \dots, t$, then $R = R_1 + R_2 + \cdots + R_t$ satisfies $r\text{-Bnd}(k)$, where $k = \max\{k_i \mid i = 1, 2, \dots, t\}$.*

(ii) *Let S be a multiplicatively closed set of central nondivisors of zero of the ring R . If R satisfies $r\text{-Bnd}(k)$, then the localization R_S also satisfies $r\text{-Bnd}(k)$.*

PROOF. (i) is easy and (ii) follows from the fact that $I = (I \cap R)R_S$ for each right ideal I of R_S . $\square$

LEMMA 3. *Let $\mathfrak{p}$ be a prime ideal of the commutative Noetherian ring R . If $R/\mathfrak{p}$ satisfies Bnd then $R/\mathfrak{p}^s$ satisfies Bnd for all $s \in \mathbb{Z}^+$.*

PROOF. Say $R/\mathfrak{p}$ satisfies $\text{Bnd}(k)$. The proof is by induction on s . Say $R/\mathfrak{p}^{s-1}$ satisfies $\text{Bnd}(k_{s-1})$. The $R/\mathfrak{p}$ -module $\mathfrak{p}^{s-1}/\mathfrak{p}^s$ is generated by, say, m elements. Thus any submodule of $\mathfrak{p}^{s-1}/\mathfrak{p}^s$ can be generated by $m \cdot k$ elements. Hence any ideal of $R/\mathfrak{p}^s$ can be generated by $k_{s-1} + m \cdot k$ elements and $R/\mathfrak{p}^s$ satisfies Bnd . $\square$

The following important theorem of O. Forster and R. G. Swan plays a crucial role in the proof of our result.

THEOREM. *Let R be a commutative Noetherian ring and S an R -algebra which is finitely generated as an R -module. Assume the Krull dimension of R is d , and let M be a finitely generated right S -module. If for each maximal ideal $\mathfrak{m}$ of R , $M_{\mathfrak{m}}$ is generated by r elements over $S_{\mathfrak{m}}$, then M can be generated over S by $r + d$ elements.*

PROOF. See Swan [12]. $\square$

3. Theorem.

THEOREM 1. *Let $R \subseteq S$ be rings with $R \subseteq Z(S)$. If S is a finitely generated R -module then R satisfies Bnd if and only if S satisfies $r\text{-Bnd}$.*

PROOF. If R satisfies Bnd , then S satisfies $r\text{-Bnd}$ and $l\text{-Bnd}$ by Lemma 1(iii), since S is a finitely generated R -module.

Conversely, suppose S satisfies $r\text{-Bnd}(b_0)$. Since S is finitely generated as an R -module, R is Noetherian by Eisenbud's generalization of Eakin's theorem [4]. Since S is right and left Noetherian, S has Krull dimension at most one by Stafford's theorem [11], and thus, since S is integral over R , R has Krull dimension at most one, by the "Going-Up" theorem [1].

Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_d$ be the minimal primes of the commutative Noetherian ring R . Then there exists a positive integer s such that $0 = \mathfrak{p}_1^s \cap \mathfrak{p}_2^s \cap \dots \cap \mathfrak{p}_d^s$. If we can show that each $R/\mathfrak{p}_i$ satisfies Bnd, then, since the sequence of R -modules

$$0 \rightarrow R \rightarrow R/\mathfrak{p}_1^s + \dots + R/\mathfrak{p}_d^s$$

is exact, the theorem will follow from Lemma 3. Let $\mathfrak{p}$ be a minimal prime of R . Since S is integral over R , there exists a prime ideal P of S such that $P \cap R = \mathfrak{p}$. Hence S/P is a finitely generated $R/\mathfrak{p}$ -module, $R/\mathfrak{p} \subseteq Z(S/P)$ and S/P satisfies r -Bnd. Thus we may assume that S is a prime ring and that R is a central integral domain contained in S .

We utilize a clever technique of Robson and Small [7] to produce an “intermediate” ring T such that

- (i) $R \subseteq T \subseteq S$,
- (ii) There exists a nonzero ideal A of S such that $A \subseteq T$,
- (iii) $R \subseteq T \subseteq M_n(R)$ for some positive integer n .

T is manufactured as follows: Say $S = a_1R + a_2R + \dots + a_tR$. Choose a maximal R -independent subset of the a_i . After renaming, if necessary, we may assume that $F = a_1R + a_2R + \dots + a_nR$ is a free R -module, and $a_jR \cap F \neq 0$ for $j > n$. Let $B_j = \{r \in R \mid a_jr \in F\}$. Notice that $B_j \neq 0$ and let $B = \bigcap_{j=1}^t B_j \neq 0$. Also $SB = BS = \sum_{i=1}^t a_iB \subseteq \sum_{i=1}^t a_iB_i \subseteq F$. Set $A = BS$, and let $T = R + A$. Since $TF = (R + A)F \subseteq F + A \subseteq F$, F is a T -module. Since $A \subseteq F$, and A is an ideal of the prime ring S , we have that F is a faithful T -module. Thus there is an embedding $T \rightarrow \text{End}_R(F) = M_n(R)$. When this embedding is restricted to R it is canonical.

Since S is finitely generated as a module over a commutative ring, S satisfies a polynomial identity and thus every ideal of the prime ring S intersects $Z(S)$ nontrivially. (See, for example, Rowen [8].) Let $0 \neq A_0 = A \cap Z(S) \subseteq T \cap Z(S) = D$. Since $R \subseteq D \subseteq S$, S is a finitely generated D -module, and thus, as before, D is a Noetherian domain whose Krull dimension is at most one. If D has Krull dimension zero, we are done since R would be a field. So we may assume D has Krull dimension one. There are only finitely many maximal ideals of D which contain A_0 . Let $\mathfrak{m}$ be a maximal ideal of D which contains A_0 . Now $D_{\mathfrak{m}}$ is a local Noetherian domain of Krull dimension one, and thus satisfies Bnd by Cohen’s theorem [3]. Since T is a finitely generated module over D , $T_{\mathfrak{m}}$ satisfies r -Bnd($b_{\mathfrak{m}}$) for some positive integer $b_{\mathfrak{m}}$. Let $b_1 = \max\{b_{\mathfrak{m}} \mid \mathfrak{m} \text{ is a maximal ideal of } D \text{ containing } A_0\}$. Thus $T_{\mathfrak{m}}$ satisfies r -Bnd(b_1) for all maximal ideals $\mathfrak{m}$ of D containing A_0 . Next let $\mathfrak{m}$ be a maximal ideal of D which does not contain A_0 . Then $A_{\mathfrak{m}} = T_{\mathfrak{m}} = S_{\mathfrak{m}}$ since $1 \in A_{\mathfrak{m}}$. Since $S_{\mathfrak{m}}$ satisfies r -Bnd(b_0), for each right ideal I of T , $I_{\mathfrak{m}}$ can be generated as a right $S_{\mathfrak{m}}$ -module by b_0 elements, and so $I_{\mathfrak{m}}$ can be generated as a $T_{\mathfrak{m}}$ -module by b_0 elements. Thus $T_{\mathfrak{m}}$ satisfies r -Bnd(b_0) for all maximal ideals $\mathfrak{m}$ of D not containing A_0 . Let $b = \max\{b_0, b_1\}$. For each right ideal I of T , $I_{\mathfrak{m}}$ can be generated as a right $T_{\mathfrak{m}}$ -module by b elements. By the Forster-Swan theorem, T satisfies r -Bnd($b + 1$).

Since $R \subseteq T \subseteq M_n(R)$, $M_n(R)$ is a finitely generated right T -module and thus

$M_n(R)$ satisfies $r\text{-Bnd}(c)$, for some positive integer c . Let I be any ideal of R . since

$$\begin{bmatrix} I & I & \cdots & I \\ 0 & 0 & \cdots & 0 \\ & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

is a right ideal of $M_n(R)$, it can be generated by c elements. Clearly I can be generated by $c \cdot n$ elements and R satisfies Bnd . $\square$

It is natural to ask whether we can relax the condition that S is a finitely generated R -module and replace it by the weaker condition that S is integral over $R \subseteq Z(S)$ or that S satisfies a polynomial identity. In the last section of this paper we show that these generalizations fail.

4. Affine PI rings. A ring S is said to be ring finite over R if S is finitely generated as an algebra over a commutative ring $R \subseteq Z(S)$. We usually denote this by $S = R\{a_1, a_2, \dots, a_d\}$. If R is a field then we say that S is affine over R . The following theorems contain, as special cases, the theorem on affine PI rings mentioned in the introduction. We begin by stating a theorem of Sarraillé which plays a crucial role in the proof of our result.

THEOREM. *let S be a semiprime ring of Krull dimension at most one which satisfies a polynomial identity. Suppose that S is ring finite over $R \subseteq Z(S)$ and that S is right and left Noetherian. Then S is a finitely generated module over $Z(S)$.*

PROOF. See Theorem 2.18 of Sarraillé [10].

THEOREM 2. *Let $S = R\{a_1, a_2, \dots, a_d\}$ be ring finite over $R \subseteq Z(S)$. Suppose that S is semiprime and satisfies a polynomial identity. If S satisfies $r\text{-Bnd}$, then S also satisfies $l\text{-Bnd}$.*

PROOF. S is right Noetherian and hence it is left Noetherian by Cauchon's theorem [2]. Stafford's theorem [11] says that S has Krull dimension at most one. An application of Sarraillé's theorem yields that S is module finite over its center $Z = Z(S)$. Theorem 1 says that Z satisfies Bnd and thus S satisfies $l\text{-Bnd}$. $\square$

By using standard ring-theoretic techniques it is now possible to replace the semiprime hypothesis in Theorem 2 by assuming the weaker condition that S is Noetherian on both sides.

THEOREM 3. *Let $S = R\{a_1, a_2, \dots, a_d\}$ be ring finite over $R \subseteq Z(S)$. Suppose that S is right and left Noetherian and satisfies a polynomial identity. If S satisfies $r\text{-Bnd}$, then S also satisfies $l\text{-Bnd}$.*

PROOF. Let N be the prime radical of S , and let t be a positive integer such that $N^t = 0$. By Theorem 2, $\bar{S} = S/N$ satisfies $l\text{-Bnd}$. For $i = 1, 2, \dots, t-1$ we have that N^i/N^{i+1} is a finitely generated left $\bar{S}$ -module, and hence N^i/N^{i+1} satisfies $l\text{-Bnd}$ as a left $\bar{S}$ -module and thus as a left S -module. Since $(N^i/N^{i+2})/(N^{i+1}/N^{i+2}) \simeq N^i/N^{i+1}$, repeated application of Lemma 1(ii) implies that N^i/N^{i+j} satisfies $l\text{-Bnd}$ as an S -module for $j \geq 1$. In particular, $N \simeq N/N^t$ satisfies $l\text{-Bnd}$ as an S -module. Since N and S/N satisfy $l\text{-Bnd}$, we have that S satisfies $l\text{-Bnd}$. $\square$

5. Examples. The first example is due to Wadsworth and Schelter (see Sarraillé [10]). We will construct a prime ring S which satisfies both r -Bnd and l -Bnd as well as a polynomial identity, but for which $Z = Z(S)$ does not satisfy Bnd. This and the next example show that the natural generalizations of Theorem 1 fail.

EXAMPLE 1. Let F_1 and F_2 be subfields of the field E such that E is a finite extension of F_1 and F_2 , but E is an infinite extension of $F = F_1 \cap F_2$. Set

$$S = \begin{bmatrix} F_1 + xE[x] & xE[x] \\ xE[x] & F_2 + xE[x] \end{bmatrix}$$

and

$$R = \left\{ \begin{bmatrix} f(x) & 0 \\ 0 & f(x) \end{bmatrix} \mid f(x) \in F + xE[x] \right\},$$

where x is an indeterminate. Since $M_2(xE[x])$ is an ideal of both S and $M_2(E[x])$, and since $M_2(E[x])$ is a prime ring, we have that S is a prime ring. Since $S \subseteq M_2(E[x])$, S satisfies a polynomial identity. For $i = 1, 2$, $F_i + xE[x]$ satisfies Bnd since $E[x]$ is a finitely generated module over $F_i + xE[x]$. Thus

$$T = \begin{bmatrix} F_1 + xE[x] & 0 \\ 0 & F_2 + xE[x] \end{bmatrix} \text{ satisfies Bnd.}$$

Since S is finitely generated as a right and left module over T , S satisfies r -Bnd and l -Bnd. Finally, $R = Z(S)$ and R does not satisfy Bnd since R is not even Noetherian. $\square$

Our next example is a ring S which satisfies r -Bnd and l -Bnd and is integral over its center R which is not Noetherian. This example is taken from Blair [1] and is based on a commutative ring constructed by Mollier [6].

EXAMPLE 2. Mollier has constructed a commutative ring $B = k[T]/(T^2)$ where k is a field of characteristic two such that the degree of k over k^2 , the subfield of squares, is infinite. B is equipped with an automorphism σ of order two such that $B^\sigma = \{b \in B \mid \sigma(b) = b\} = k^2 + kt$, where $t = T + (T^2)$, is a non-Noetherian subring of B . Also note that B is a two-dimensional algebra over the field k .

Form the skew polynomial ring $B[X, \sigma]$ where multiplication is given by $Xb = \sigma(b)X$ for $b \in B$. Let $S = B[X, \sigma]/(X^2)$ and let x be the image of X modulo (X^2) . S is a finitely generated right and left module over B and hence over k . Thus S satisfies r -Bnd and l -Bnd. Let $A = \{b \in B \mid bc = b\sigma(c) \text{ for all } c \in B\}$. Then $R = Z(S) = \{b_0 + b_1x \mid b_0 \in B^\sigma, b_1 \in A\}$. If I is an ideal of B^σ then $I^* = \{b_0 + b_1x \mid b_0 \in I, b_1 \in A\}$ is an ideal of R , and so R is not Noetherian and hence does not satisfy Bnd. If $s = b_0 + b_1x \in S$, then s satisfies $X^2 - (b_0 + \sigma(b_0))X + b_0\sigma(b_0) = 0$, while $b_0 + \sigma(b_0)$ and $b_0\sigma(b_0)$ are elements of $B^\sigma \subseteq R$. Finally, one can also check that S satisfies the standard identity of degree 4. $\square$

The next example is an affine PI ring which satisfies r -Bnd but not l -Bnd. This shows that the hypotheses of Theorems 2 and 3 are needed.

EXAMPLE 3. Let F be a field and set

$$S = \begin{bmatrix} F & F[x] \\ 0 & F[x] \end{bmatrix}$$

where x is an indeterminate. Clearly

$$T = \begin{bmatrix} F & 0 \\ 0 & F[x] \end{bmatrix}$$

satisfies Bnd and since S is a finitely generated right T -module, S satisfies r -Bnd. On the other hand S fails to satisfy l -Bnd since it is not left Noetherian.

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