NOTE ON NILPOTENT DERIVATIONS

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ABSTRACT. Let R be a prime ring with center Z . Suppose that d is a derivation on R such that $d^n(x) \in Z$ for all x , where n is a fixed integer. It is shown that either $d^n(x) = 0$ for all $x \in R$ or R is a commutative integral domain. Moreover, the same conclusion holds even if we assume that $d^n(x) \in Z$ merely for all $x \in I$, where I is a nonzero ideal of R .

Let R be a prime ring with center Z and a derivation d . Suppose that $d(x) \in Z$ for all $x \in R$, then one can easily show that either $d = 0$ or R is commutative [3, Proof, Theorem 2]. If $d^2(x) \in Z$ for all $x \in R$, we have the same conclusion when R is not of characteristic 2 [3, Theorem 3]. On the other hand, in case $\text{char } R = 2$ there exist nonzero derivations d on R such that $d^2 = 0$ provided R is not a domain. One might naturally ask what we can conclude in general if $d^n(x) \in Z$ for all $x \in R$, where n is some fixed integer. Certainly, this condition is fulfilled when d is nilpotent of nilpotency n or when R is a commutative ring. We prove here that these are indeed the only two possibilities where $d^n(R) \subseteq Z$ can happen.

THEOREM 1. *Let R be a prime ring with center Z and d a derivation on R such that $d^n(R) \subseteq Z$ for some natural number n . Then either $d^n = 0$ or R is commutative.*

PROOF. First note that if $Z = 0$ there is nothing to prove. So we assume that $Z \neq 0$ and proceed to prove by induction on n . When $n = 1$, the conclusion holds as we mentioned at the beginning. Now assume that $n > 1$ and the theorem is true for $< n$.

Suppose that $\text{char } R = p$; then $\delta = d^p$ is also a derivation on R . If n is divisible by p , the assumption reads $\delta^{n/p}(R) \subseteq Z$. Since $n/p < n$, it follows from induction hypothesis that either $d^n = \delta^{n/p} = 0$ or R is commutative. So we may henceforth assume that R is of characteristic 0 or a prime p not dividing n .

For $\alpha \in Z$ and $x \in R$, we have $d^n(\alpha d^{n-1}(x)) \in Z$. That is,

$$d^n(\alpha) d^{n-1}(x) + \sum_{i=1}^n \binom{n}{i} d^{n-i}(\alpha) d^{n+i-1}(x) \in Z$$

for all $\alpha \in Z$, $x \in R$. Note that each term in the summation is already in Z , therefore $d^n(Z) d^{n-1}(R) \subseteq Z$. Consequently, either $d^n(Z) = 0$ or $d^{n-1}(R) \subseteq Z$. If $d^{n-1}(R) \subseteq Z$ we are done by induction hypothesis. So we assume that $d^n(Z) = 0$.

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Let m be the smallest integer such that $d^m(Z) = 0$. Assume first that $m \geq 2$. For $\alpha \in Z$ and $x \in R$ we have

$$d^n(d^{m-2}(\alpha)x) = d^{m-2}(\alpha)d^n(x) + nd^{m-1}(\alpha)d^{n-1}(x) \in Z.$$

Since $d^{m-2}(\alpha)d^n(x) \in Z$, we have $nd^{m-1}(\alpha)d^{n-1}(x) \in Z$ for all $\alpha \in Z$, $x \in R$. Recall that $\text{char } R$ is 0 or $p \nmid n$, so $d^{m-1}(Z)d^{n-1}(R) \subseteq Z$ follows. By the minimality of m , $d^{m-1}(Z) \neq 0$ so $d^{n-1}(R) \subseteq Z$, and we are done. Hence it remains to check the case when $m = 1$, that is, $d(Z) = 0$.

Since $Z \neq 0$, we may localize R at Z to get a ring $S = \{a/\alpha \mid a \in R, \alpha \in Z, \alpha \neq 0\}$. Then S is a prime ring with center the quotient field of Z . Moreover, we can extend d to a derivation $\bar{d}$ on S by defining $\bar{d}(a/\alpha) = d(a)/\alpha$. Then $\bar{d}^n(S) \subseteq Z(S)$, the center of S . So we may replace R and d by S and $\bar{d}$, respectively, and assume that Z is a field. Thus $d^n(R)$ is a Z -subspace of dimension at most 1.

Let k be the smallest integer such that $d^k(R)$ is finite dimensional over Z . Assume that $k \geq 1$. For $x, y \in R$ we have

$$d^n(d^{k-1}(x)y) = d^{k-1}(x)d^n(y) + \sum_{i=1}^n \binom{n}{i} d^{k+i-1}(x)d^{n-i}(y) \in Z.$$

Note that $d^{k+i-1}(R)$ is finite dimensional over Z , so is $d^{k+i-1}(R)d^{n-i}(y)$ for all $y \in R$ and $i \geq 1$. Thus it follows that $d^{k-1}(R)d^n(y)$ must be finite dimensional over Z for each $y \in R$. Since $d^n(y) \in Z$, we would have $d^{k-1}(R)$ finite dimensional to contradict the choice of k provided $d^n(y)$ were not zero. Hence we must have $d^n(R) = 0$.

At last, assume that $k = 0$, that is, R is finite dimensional over Z . Being a prime ring, R must be simple. Now $d(Z) = 0$, so d must be inner by a classic result [2, Proposition, p. 100]. In other words, there exists $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$. Thus $ad^n(x) = d^n(ax) \in Z$, and so either $a \in Z$ or $d^n(x) = 0$ for all $x \in R$. In either case, we have always $d^n = 0$. This completes the proof.

Next we extend the previous theorem by conditioning $d^n(x) \in Z$ merely for all x in some nonzero ideal I of R .

THEOREM 2. *Let R be a prime ring with center Z and I a nonzero ideal of R . Suppose that d is a derivation on R such that $d^n(I) \subseteq Z$ for some natural number n . Then either $d^n = 0$ or R is commutative.*

PROOF. Let $J = I + d(I) + d^2(I) + \cdots$. Then J is a nonzero ideal of R , $d(J) \subseteq J$, and $d^n(J) \subseteq Z$. Applying Theorem 1, we can conclude that either $d^n(J) = 0$ or J is a commutative ring. However, if $d^n(J) = 0$, then $d^n = 0$ by a theorem due to Chung and Luh [1]; while if J is commutative, so is the whole ring R . Thus the theorem is proved.

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