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ON FUNCTIONS WHOSE DERIVATIVE HAS POSITIVE REAL PART

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ABSTRACT. Let R be the class of normalised analytic functions f, defined in the open disc D, such that $\operatorname{Re} f'(z) > 0$ for $z \in D$. For $f \in R$, a best possible growth estimate for $|zf'(z)/f(z)|$ is obtained.

1. Introduction. Denote by R the class of functions f which are regular in $D = \{z\colon |z| < 1\}$ satisfying $f(0) = 0$, $f'(0) = 1$, and $\operatorname{Re} f'(z) > 0$ for $z \in D$.

A classical paper of Alexander **[1]** shows that such functions are univalent in D. MacGregor **[2**, Theorem 1] considered the class R and developed the usual distortion and coefficient results, all of which are immediate consequences of the representation theorem for functions of positive real part **[3]**.

An omission from MacGregor's paper is a consideration of a distortion theorem for $|zf'(z)/f(z)|$. Indeed, it appears that no such result has been proved. The object of this note is to give such an estimate.

2. Results and proofs. We begin by stating MacGregor's results (loc. cit).

THEOREM A.

$$\text{(1)} \qquad \textit{Let } f \in R \textit{ and } f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \textit{for } z \in D.$$

Then for $z = re^{i\theta}$, $0 \leqslant r < 1$

$$|a_n| \leqslant 2/n, \qquad n = 2, 3, \ldots,$$
$$|f'(z)| \leqslant (1+r)/(1-r),$$
$$\operatorname{Re} f'(z) \geqslant (1-r)/(1+r),$$
$$-r + 2\log(1+r) \leqslant |f(z)| \leqslant -r - 2\log(1-r).$$

We now add the following

THEOREM 1. *Let* $f \in R$. *Then for* $z = re^{i\theta}$, $0 < r < 1$,

$$\left|\frac{zf'(z)}{f(z)}\right| < \frac{K}{(1-r)\log(1-r)^{-1}},$$

where K is an absolute constant. This growth rate is best possible.

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PROOF. Since $\operatorname{Re} f'(z) > 0$ for $z \in D$, we can write [3]

$$f'(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t), \tag{2}$$

where μ is increasing and $\mu(2\pi) - \mu(0) = 2\pi$. Thus,

$$|f'(z)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1 + ze^{-it}}{1 - ze^{-it}}\right| d\mu(t).$$

We now use a method of Twomey [6] to estimate the above integrand by noting that, for $r < 1$,

$$\left|\frac{1 + ze^{-it}}{1 - ze^{-it}}\right| \leqslant \frac{2r\log((1 + r)/|1 - ze^{-it}|)}{(1 - r)\log((1 + r)/(1 - r))} + 1.$$

Hence

$$|f'(z)| \leqslant \frac{2r\log(1 + r)}{(1 - r)\log((1 + r)/(1 - r))} \tag{3}$$
$$+ \frac{1}{\pi}\int_0^{2\pi} \frac{r\log|1 - ze^{-it}|^{-1}}{(1 - r)\log((1 + r)/(1 - r))}\, d\mu(t) + 1.$$

Next we note that, from (2),

$$\frac{f'(z) - 1}{z} = \frac{1}{\pi}\int_0^{2\pi} \frac{e^{-it}}{1 - ze^{-it}}\, d\mu(t),$$

and so

$$F(z) = \frac{1}{\pi}\int_0^{2\pi} \log\frac{1}{1 - ze^{-it}}\, d\mu(t),$$

where

$$F(z) = \int_0^z \frac{f'(\zeta) - 1}{\zeta}\, d\zeta. \tag{4}$$

Thus,

$$\operatorname{Re} F(z) = \frac{1}{\pi}\int_0^{2\pi} \log\frac{1}{|1 - ze^{-it}|}\, d\mu(t). \tag{5}$$

From (3) and (5) we obtain

$$|f'(z)| \leqslant \frac{2r\log(1 + r)}{(1 - r)\log((1 + r)/(1 - r))} + \frac{|zF(z)|}{(1 - r)\log((1 + r)/(1 - r))} + 1. \tag{6}$$

Now (1) and (4) gives

$$zF(z) = f(z) - z + \sum_{n=2}^{\infty} \frac{a_n}{n - 1} z^n,$$

and so using the coefficient estimate in Theorem A we have

$$|zF(z)| \leqslant |f(z)| + 3r + (2/r)(1 - r)\log(1 - r)^{-1}. \tag{7}$$

Finally, from (6), (7) and the lower bound for $|f(z)|$ in Theorem A we obtain Theorem 1.

The function $f(z) = -z + 2\log(1 - z)^{-1}$ shows that the growth rate in Theorem 1 is best possible.

We remark that a sharp estimate in Theorem 1 remains an open question.

An immediate consequence of Theorem 1 is that if $f \in R$ and is bounded, then

$$M(r, f') = O(1)\left[(1 - r)\log(1 - r)^{-1}\right]^{-1}.$$

Following Twomey [5], we have, in the opposite direction,

THEOREM 2. *There exists a bounded $f \in R$ such that*

$$\limsup_{r\to 1} M(r, f')(1 - r)\log\frac{1}{1 - r} > 0. \tag{8}$$

PROOF. The examples given in [5, §9] will suffice on writing

$$f'(z) = \sum_{n=1}^{\infty} \lambda_n \frac{1 + r_n e^{-i\theta_n} z}{1 - r_n e^{-i\theta_n} z}$$

for $z \in D$, with $\{r_n\}$, $\{\theta_n\}$ and $\{\lambda_n\}$ defined as in [5, §9]. It is then clear that (8) holds and that f is bounded.

REMARK. Theorem 1 can also be proved using the less elementary method of Ruschweyh [4], but this also appears not to give the sharp result.

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