A NOTE ON TWO SUMMABILITY METHODS

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ABSTRACT. In this paper we have established a relation between the $|\bar{N}, p_n|_k$ and $|C, 1|_k$ summability methods, which is the converse of the result of Bor [2].

1. Introduction. Let $\sum a_n$ be a given infinite series with partial sums s_n , and $u_n = na_n$. By σ_n^α and t_n^α we denote the n th Cesàro means of order α ($\alpha > -1$) of the sequences (s_n) and (u_n) , respectively. The series $\sum a_n$ is said to be absolutely summable (C, α) with index k , or simply summable $|C, \alpha|_k$, $k \geq 1$, if

$$(1.1) \quad \sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty \quad [3].$$

Since $t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$ (see [4]), condition (1.1) can also be written as

$$(1.2) \quad \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty.$$

Let (p_n) be a sequence of positive real constants such that

$$(1.3) \quad P_n = \sum_{v=0}^n p_v \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad (P_{-1} = p_{-1} = 0).$$

The sequence-to-sequence transformation

$$(1.4) \quad T_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v$$

defines the sequence (T_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n) , generated by the sequence of coefficients (p_n) . The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if

$$(1.5) \quad \sum_{n=1}^{\infty} \left[\frac{P_n}{P_{n-1}} \right]^{k-1} |T_n - T_{n-1}|^k < \infty \quad [1].$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\bar{N}, p_n|$) summability.

2. It is known that the summability $|\bar{N}, p_n|_k$ and the summability $|C, 1|_k$ are, in general, independent of each other provided that p_n must differ from one for at least one $n \in N$. In a recent paper Bor [2] has proved the following theorem.

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THEOREM A. Let (p_n) be a sequence of positive real constants such that, as $n \rightarrow \infty$,

$$(2.1) \quad (i) \quad np_n = \mathcal{O}(P_n), \quad (ii) \quad P_n = \mathcal{O}(np_n).$$

If Σa_n is summable $|C, 1|_k$, then it is also summable $|\bar{N}, p_n|_k$, $k \geq 1$. Notice that, to see the hypotheses (i) and (ii) in the theorem are satisfied by at least one $p_n \neq 1$, it is sufficient to take $p_n = n$ for all $n \in N$. In this paper we shall prove the following theorem, which is the converse of Theorem A.

THEOREM 1. Let (p_n) be a sequence of positive real constants such that it satisfies the condition (2.1). If Σa_n is summable $|\bar{N}, p_n|_k$, then it is also summable $|C, 1|_k$, $k \geq 1$.

3. Proof of Theorem 1. We have

$$T_n = \frac{1}{P_n} \sum_{\nu=0}^n p_\nu s_\nu = \frac{1}{P_n} \sum_{\nu=0}^n (P_n - P_{\nu-1}) a_\nu.$$

The series Σa_n summable $|\bar{N}, p_n|_k$ means that

$$(3.1) \quad \sum_{n=1}^{\infty} \left[\frac{P_n}{p_n} \right]^{k-1} |\Delta T_{n-1}|^k < \infty,$$

where

$$\Delta T_{n-1} = \left[\frac{-1}{P_{n-1}} + \frac{1}{P_n} \right] \sum_{\nu=0}^n P_{\nu-1} a_\nu = -\frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^n P_{\nu-1} a_\nu.$$

So

$$P_{n-1} a_n = -\frac{P_n P_{n-1}}{p_n} \Delta T_{n-1} + \frac{P_{n-1} P_{n-2}}{p_{n-1}} \Delta T_{n-2}.$$

That is,

$$(3.2) \quad a_n = -\frac{P_n}{p_n} \Delta T_{n-1} + \frac{P_{n-2}}{p_{n-1}} \Delta T_{n-2}.$$

It is easily verified that this holds also when $n = 1$ (since in this case $P_{n-2} = 0$). By t_n we denote the n th $(C, 1)$ mean of the sequence (na_n) .

$$t_n = \frac{1}{n+1} \sum_{\nu=1}^n \nu a_\nu.$$

Since, $a_\nu = -(P_\nu/p_\nu) \Delta T_{\nu-1} + (P_{\nu-2}/p_{\nu-1}) \Delta T_{\nu-2}$, by (3.2), we have

$$\begin{aligned} t_n &= \frac{1}{n+1} \sum_{\nu=1}^n \nu \left\{ -\frac{P_\nu}{p_\nu} \Delta T_{\nu-1} + \frac{P_{\nu-2}}{p_{\nu-1}} \Delta T_{\nu-2} \right\} \\ &= \frac{1}{n+1} \sum_{\nu=1}^{n-1} \{-\nu\} \frac{P_\nu}{p_\nu} \Delta T_{\nu-1} - \frac{n P_n}{(n+1) p_n} \Delta T_{n-1} + \frac{1}{n+1} \sum_{\nu=1}^n \nu \frac{P_{\nu-2}}{p_{\nu-1}} \Delta T_{\nu-2} \\ &= \frac{1}{n+1} \sum_{\nu=1}^{n-1} (-\nu) \frac{P_\nu}{p_\nu} \Delta T_{\nu-1} + \frac{1}{n+1} \sum_{\nu=1}^{n-1} (\nu+1) \frac{P_{\nu-1}}{p_\nu} \Delta T_{\nu-1} - \frac{n P_n}{(n+1) p_n} \Delta T_{n-1} \\ &= \frac{1}{n+1} \left\{ \sum_{\nu=1}^{n-1} \frac{1}{p_\nu} \Delta T_{\nu-1} [-\nu P_\nu + (\nu+1) P_{\nu-1}] \right\} - \frac{n P_n}{(n+1) p_n} \Delta T_{n-1}. \end{aligned}$$

Since $-\nu P_\nu + (\nu + 1)P_{\nu-1} = P_\nu - (\nu + 1)p_\nu$, we have

$$\begin{aligned} t_n &= \frac{1}{n+1} \left\{ \sum_{\nu=1}^{n-1} \frac{1}{p_\nu} \Delta T_{\nu-1} [P_\nu - (\nu + 1)p_\nu] \right\} - \frac{nP_n}{(n+1)p_n} \Delta T_{n-1} \\ &= \frac{1}{n+1} \sum_{\nu=1}^{n-1} \frac{P_\nu}{p_\nu} \Delta T_{\nu-1} - \frac{1}{n+1} \sum_{\nu=1}^{n-1} (\nu + 1) \Delta T_{\nu-1} - \frac{nP_n}{(n+1)p_n} \Delta T_{n-1} \\ &= t_{n,1} + t_{n,2} + t_{n,3}, \quad \text{say.} \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_{n,r}|^k < \infty \quad \text{for } r = 1, 2, 3.$$

Now, applying Hölder's inequality, we have

$$\begin{aligned} \sum_{n=2}^{m+1} \frac{1}{n} |t_{n,1}|^k &= \sum_{n=2}^{m+1} \frac{1}{n} \left| \frac{1}{n+1} \sum_{\nu=1}^{n-1} \frac{P_\nu}{p_\nu} \Delta T_{\nu-1} \right|^k \\ &\leq \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} \left\{ \sum_{\nu=1}^{n-1} \frac{P_\nu}{p_\nu} |\Delta T_{\nu-1}| \right\}^k \leq \sum_{n=2}^{m+1} \frac{1}{n^2} \sum_{\nu=1}^{n-1} \left[\frac{P_\nu}{p_\nu} \right]^k |\Delta T_{\nu-1}|^k \times \left\{ \frac{1}{n} \sum_{\nu=1}^{n-1} 1 \right\}^{k-1} \\ &= \mathcal{O}(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \sum_{\nu=1}^{n-1} \left[\frac{P_\nu}{p_\nu} \right]^k |\Delta T_{\nu-1}|^k = \mathcal{O}(1) \sum_{\nu=1}^m \left[\frac{P_\nu}{p_\nu} \right]^k |\Delta T_{\nu-1}|^k \sum_{n=\nu+1}^{m+1} \frac{1}{n^2} \\ &= \mathcal{O}(1) \sum_{\nu=1}^m \left[\frac{P_\nu}{p_\nu} \right]^k \frac{1}{\nu} |\Delta T_{\nu-1}|^k. \end{aligned}$$

Since $1/\nu = \mathcal{O}[p_\nu/P_\nu]$ by (2.1(ii)), we have

$$\sum_{n=2}^{m+1} \frac{1}{n} |t_{n,1}|^k = \mathcal{O}(1) \sum_{\nu=1}^m \left[\frac{P_\nu}{p_\nu} \right]^{k-1} |\Delta T_{\nu-1}|^k = \mathcal{O}(1) \quad \text{as } m \rightarrow \infty$$

by (3.1). Again,

$$\begin{aligned} \sum_{n=2}^{m+1} \frac{1}{n} |t_{n,2}|^k &= \sum_{n=2}^{m+1} \frac{1}{n} \left| \frac{1}{n+1} \sum_{\nu=1}^{n-1} (\nu + 1) \Delta T_{\nu-1} \right|^k \\ &= \sum_{n=2}^{m+1} \frac{1}{n} \left| \frac{1}{n+1} \sum_{\nu=1}^{n-1} \frac{\nu+1}{\nu} \nu \Delta T_{\nu-1} \right|^k \\ &= \mathcal{O}(1) \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} \left\{ \sum_{\nu=1}^{n-1} \nu |\Delta T_{\nu-1}| \right\}^k \\ &= \mathcal{O}(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \sum_{\nu=1}^{n-1} \nu^k |\Delta T_{\nu-1}|^k \times \left\{ \frac{1}{n} \sum_{\nu=1}^{n-1} 1 \right\}^{k-1} \\ &= \mathcal{O}(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \sum_{\nu=1}^{n-1} \nu^k |\Delta T_{\nu-1}|^k = \mathcal{O}(1) \sum_{\nu=1}^m \nu^k |\Delta T_{\nu-1}|^k \sum_{n=\nu+1}^{m+1} \frac{1}{n^2} \\ &= \mathcal{O}(1) \sum_{\nu=1}^m \nu^k |\Delta T_{\nu-1}|^k \frac{1}{\nu} = \mathcal{O}(1) \sum_{\nu=1}^m \nu^{k-1} |\Delta T_{\nu-1}|^k. \end{aligned}$$

Since $\nu = \mathcal{O}[P_\nu/p_\nu]$ by (2.1(i)), we have

$$\sum_{n=1}^{m+1} \frac{1}{n} |t_{n,2}|^k = \mathcal{O}(1) \sum_{\nu=1}^m \left[\frac{P_\nu}{p_\nu} \right]^{k-1} |\Delta T_{\nu-1}|^k = \mathcal{O}(1) \quad \text{as } m \rightarrow \infty$$

by (3.1). Finally, we have

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_{n,3}|^k = \sum_{n=1}^m \frac{1}{n} \left| \frac{nP_n}{(n+1)p_n} \Delta T_{n-1} \right|^k = \mathcal{O}(1) \sum_{n=1}^m \frac{1}{n} \left[\frac{P_n}{p_n} \right]^k |\Delta T_{n-1}|^k.$$

Since $1/n = \mathcal{O}[p_n/P_n]$ by (2.1(ii)), we have

$$\sum_{n=1}^m \frac{1}{n} |t_{n,3}|^k = \mathcal{O}(1) \sum_{n=1}^m \left[\frac{P_n}{p_n} \right]^{k-1} |\Delta T_{n-1}|^k = \mathcal{O}(1) \quad \text{as } m \rightarrow \infty$$

by (3.1). Therefore, we get

$$\sum_{n=1}^m \frac{1}{n} |t_{n,r}|^k = \mathcal{O}(1) \quad \text{as } m \rightarrow \infty, \text{ for } r = 1, 2, 3.$$

This completes the proof of the theorem.

If we put these two results together we have the following theorem.

THEOREM 2. Suppose (p_n) is a sequence of nonnegative real constants such that $P_n = \sum_{\nu=0}^n p_\nu \neq 0$, $P_n \rightarrow \infty$ as $n \rightarrow \infty$, and that (2.1) holds. Then summability $|C, 1|_k$ is equivalent to summability $|\bar{N}, p_n|_k$, $k \geq 1$.

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REFERENCES

1. H. Bor, On $|\bar{N}, p_n|_k$ summability factors of infinite series, J. Univ. Kuwait Sci. **10** (1983), 37–42.
2. ———, On two summability methods, Math. Proc. Cambridge Philos. Soc. **97** (1985), 147–149.
3. T. M. Flett, On an extension absolute summability and theorems of Littlewood and Paley, Proc. London Math. Soc. **7** (1957), 113–141.
4. E. Kogbetliantz, Sur les séries absolument sommables par la méthode des moyennes arithmétiques, Bull. Sci. Math. **9** (1925), 234–256.

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