ON THE GENERIC NONEXISTENCE OF FIRST INTEGRALS

MIKE HURLEY

ABSTRACT. The property of having no C^n first integrals other than constants is shown to be generic in $\text{Diff}^r(M)$ for each $r = 1, 2, \dots$, where n is the dimension of M .

Let M be a compact Riemannian manifold, and consider the space $\text{Diff}^r(M)$ of the C^r diffeomorphisms from M to itself with the C^r topology. Peixoto has shown in [2] that the property of having no smooth first integrals other than constants is C^1 generic. This note describes how a combination of ideas of Arraut [1] and Takens [5] yields a proof that the nonexistence of (sufficiently) smooth, nonconstant first integrals is C^r generic for any $r = 1, 2, \dots, \infty$.

If $f: M \rightarrow M$ is a diffeomorphism, then a map $g: M \rightarrow R$ is a *first integral* of f if (i) g is constant along f -orbits: $g(f(x)) = g(x)$ for all x in M , and (ii) g is not a constant. Peixoto's proof in [2] that the nonexistence of sufficiently smooth first integrals is a C^1 generic property is based on three observations. First, Sard's theorem implies that if g is sufficiently smooth (C^n where n is the dimension of M), and has no regular values in its image, then g is constant. Second, by Pugh's general density theorem, the property of having the nonwandering set equal to the closure of the hyperbolic periodic points is generic in $\text{Diff}^1(M)$ [3]. This is why Peixoto's argument works only to establish C^1 genericity. Third, if g is a C^1 first integral of f and p is a hyperbolic periodic point of f , then p is necessarily a critical point of g . Peixoto establishes this using a geometric argument based on the stable manifold theorem; a more elementary proof can be found in [4]. For f satisfying Pugh's genericity condition, it follows from continuity that every nonwandering point of f is a critical point of g . Since M is compact, every level set of g contains nonwandering points, so g has no regular values in its image, and thus is constant.

By combining a theorem of Takens and an argument due to Arraut we can avoid using Pugh's theorem and so obtain the following result:

THEOREM. *Suppose M is a compact manifold of dimension n . For each $r = 1, 2, \dots, \infty$, there is a residual subset S_r of $\text{Diff}^r(M)$ with the property that if f is in S_r , then no C^r first integral of f has any regular values in its image. In particular, if $r \geq n$, then no f in S_r has any nonconstant C^r first integrals.*

Received by the editors September 5, 1985.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 58F35.

Key words and phrases. First integral, generic property.

©1986 American Mathematical Society
0002-9939/86 \$1.00 + \$.25 per page

PROOF. Takens shows in [5] that there is a residual subset S_r of $\text{Diff}'(M)$ with the property that if f_0 is in S_r , $\delta > 0$, and X is a nonempty, closed, f_0 -invariant subset of M , then for any diffeomorphism f sufficiently C^r close to f_0 , there is a nonempty, closed, f -invariant set Y that is within δ of X in the Hausdorff metric. In particular, if U is open in M and contains X , then any diffeomorphism C^r close enough to f_0 has a nonempty, closed, invariant set contained in U .

Next we adapt an argument from [1] to show that no map in S_r has a smooth first integral.

LEMMA. Suppose $g: M \rightarrow R$ is C^r ($r \geq 1$) and that $c = g(x)$ is a regular value of g . Let $N = g^{-1}(c)$. There is an open interval I containing c , a neighborhood U of N in M , and a C^r diffeomorphism h from U onto $N \times I$ satisfying $h(x) = (h_1(x), g(x))$, and $U = g^{-1}(I)$.

PROOF OF LEMMA. Denote the Riemannian metric on M by $\langle -, - \rangle$ and let $|-|$ be the associated norm. Take an open interval I_0 , containing c , whose closure consists entirely of regular values of g . Define $U_0 = g^{-1}(I_0)$. Consider the C^{r-1} vector field $X = \text{grad}(g)/|\text{grad}(g)|$ on U_0 . By perturbing X slightly in the C^{r-1} topology we can find a C^∞ vector field Y defined on U_0 and satisfying

$$(1) \quad \langle X(x), Y(x) \rangle > \frac{1}{2} \quad \text{for all } x \text{ in } U_0.$$

Let ψ denote the flow on U_0 induced by Y . By (1), g is strictly increasing along forward ψ -orbits, and for each q in I_0 , the level set $g^{-1}(q)$ intersects each ψ -orbit in exactly one point. Consequently, for each x in U_0 there is a uniquely defined real number $T(x)$ satisfying $g(\psi(T(x), x)) = c$. By (1), the implicit function theorem applies to show that T is C^r . Define $h: U_0 \rightarrow N \times I_0$ by $h(x) = (\psi(T(x), x), g(x))$. h is clearly C^r and it is easy to check that Dh has full rank at each point of N , so by the inverse function theorem and the compactness of N , there is an open interval I in I_0 such that h is a C^r diffeomorphism from U onto $N \times I$, where $U = g^{-1}(I)$. $\square$

Now we return to the proof of the theorem. Suppose f is in $\text{Diff}'(M)$ and that g is constant along orbits of f . Assume that g is also C^r , that c is in the image of g and is a regular value of g , so that $N = g^{-1}(c)$ is an embedded codimension one submanifold of M . Let U, I and h be as in the lemma, and select a C^∞ map $j: R \rightarrow R$ satisfying

$$(2) \quad j(t) > 0 \text{ for all } t \text{ in } I, \text{ and } j(t) = 0 \text{ otherwise.}$$

$$(3) \quad \text{id} + j \text{ is a diffeomorphism.}$$

For $1 > v > 0$ define f_v by

$$(4) \quad f_v = h^{-1} \circ (\text{id}, \text{id} + v \cdot j) \circ h \circ f \text{ on } U, \text{ and by } f_v = f \text{ otherwise.}$$

As v decreases to 0, f_v converges to f in the C^r topology, and g is strictly increasing along all f_v orbits ($v > 0$) in U . If $I = (a, b)$, then it is easy to check that if x is in U and $v > 0$, then $(f_v)^k(x)$ tends to $g^{-1}(b)$ as k goes to infinity, and to $g^{-1}(a)$ as k tends to negative infinity. Consequently, U contains no nonempty, closed, f_v -invariant sets for any $1 > v > 0$. Since U is an open neighborhood of the nonempty, closed, f -invariant set N , it follows that f is not contained in Takens' residual set S_r . $\square$

REFERENCES

1. J. L. Arraut, *Note on structural stability*, Bull. Amer. Math. Soc. **72** (1966), 542–544.
2. M. M. Peixoto, *Qualitative theory of differential equations and structural stability*, Differential Equations and Dynamical Systems (J. Hale and J. LaSalle, eds.), Academic Press, New York, 1967, pp. 469–480.
3. C. Pugh, *An improved closing lemma and a general density theorem*, Amer. J. Math. **89** (1967), 1010–1021.
4. C. Robinson, *Generic properties of conservative systems*, Amer. J. Math. **92** (1970), 562–603.
5. F. Takens, *On Zeeman's tolerance stability conjecture*, Manifolds-Amsterdam 1970, Lecture Notes in Math., vol. 197, Springer-Verlag, Berlin and New York, 1971, pp. 209–219.

DEPARTMENT OF MATHEMATICS AND STATISTICS, CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO 44106