STEENROD HOMOLOGY AND LOCAL CONNECTEDNESS

JERZY DYDAK

ABSTRACT. Steenrod homology is used to explain results concerning LC^n -divisors and one-point compactifications of LC^n -spaces. It is shown that the one-point compactification wX of a locally compact metrizable space X is hlc^n iff X is hlc^n and its Steenrod k th homology is finite generated for $k \leq n$.

0. Introduction. In $[D_1]$ the author characterized LC^n -divisors ($n \geq 1$) as compacta X which are nearly 1-movable and such that $\text{pro-}H_k(X)$ is stable for $k < n$ and Mittag-Leffler (semistable in today's terminology) for $k = n$. The former condition (near 1-movability) is of homotopy nature and the latter condition is purely homological. It is well known that near 1-movability of X is needed for M/X to be LC^1 for any $M \in LC^n$ containing X (see $[Sh$ or $D_4]$). In this paper it is shown that the homology conditions are needed for M/X to be Steenrod homology locally n -connected.

Throughout the paper $\bar{H}_k(X)$ denotes the k th Steenrod homology group of X with coefficients in the group of integers (see $[M, EH$ or $BM]$.)

$\bar{H}_k^c(X)$ denotes the k th Steenrod homology with compact supports.

The author is grateful to Bob Daverman and Gerard Venema for discussions which led to Theorem 3 and its proof.

1. Locally connected spaces.

DEFINITION. A locally compact metrizable space X is said to be hlc^n if for every compact subset U of X and for every $x \in \text{int } U$ there exists a compact neighborhood V of x in U such that the inclusion induced homomorphism $\bar{H}_k(V) \rightarrow \bar{H}_k(U)$ is trivial for $k \leq n$ (see $[BM, p. 150]$).

The notion of an hlc^n -space is dual to the analogous notion of a clc^n -space which is defined using cohomology rather than homology (see $[B, p. 76]$). The following lemma is dual to the results in $[B, p. 77]$.

LEMMA 1. *For a locally compact metrizable space X the following conditions are equivalent:*

- (a) X is hlc^n .
- (b) *For each $x \in X$ and each compact neighborhood U of x there exists a compact neighborhood V of x in U such that $\text{im}(\bar{H}_k(V) \rightarrow \bar{H}_k(U))$ is countable for $k \leq n$.*

Received by the editors March 11, 1985 and, in revised form, April 24, 1985.

1980 *Mathematics Subject Classification.* Primary 55N07, 54F35; Secondary 54F43.

Key words and phrases. Steenrod homology, homology locally connected spaces.

©1986 American Mathematical Society
0002-9939/86 \$1.00 + \$.25 per page

(c) For any two compact subsets K and L of X with $K \subset \text{int } L$, $\text{im}(\bar{H}_k(U) \rightarrow \bar{H}_k(L))$ is finitely generated for $k \leq n$.

PROOF. (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (b) are trivial.

(b) $\Rightarrow$ (a) Fix $k \leq n$ and choose a decreasing sequence $U_{i+1} \subset U_i$ of compact neighborhoods of x in X such that $\{x\} = \bigcap_{i=1}^{\infty} U_i$ and $\text{im}(\bar{H}_k(U_{i+1}) \rightarrow \bar{H}_k(U_i))$ is countable for all i .

From the exact sequence ($p \geq 0$)

$$0 \rightarrow \lim^1 \tilde{H}_{p+1}(U_i) \rightarrow \tilde{H}_p(x) \rightarrow \text{inv} \lim \tilde{H}_p(U_i) \rightarrow 0$$

(see [EH, p. 270]) we get $\text{inv} \lim \tilde{H}_k(U_i) = \lim^1 \tilde{H}_k(U_i) = 0$. A result of Gray [G] says that $\{\tilde{H}_k(U_i)\}$ is semistable, and Proposition 6.1.3 of [DS, p. 77] says that $\{\tilde{H}_k(U_i)\}$ is isomorphic as a pro-group to the trivial group. Thus, for each i there exists $m > i$ with $\tilde{H}_k(U_m) \rightarrow \tilde{H}_k(U_i)$ being trivial, i.e. X is hlc^n .

(a) $\Rightarrow$ (c) This is Proposition 6.8 in [BM].

Condition (c) of Lemma 1 gives us the following (see Theorem 6.9 of [BM]).

COROLLARY 1. If $X \in \text{hlc}^n$ is compact, then $\bar{H}_k(X)$ is finitely generated for $k \leq n$.

COROLLARY 2. Suppose $f: X \rightarrow Y$ is a proper surjection of locally compact metrizable spaces. If X is hlc^n and $\tilde{H}_k(f^{-1}(y)) = 0$ for all $y \in Y$ for $k < n$, then Y is hlc^n .

PROOF. Choose compact subsets K, L of Y with $K \subset \text{int } L$. If $K' = f^{-1}(K)$, $L' = f^{-1}(L)$ then $K' \subset \text{int } L'$ and by Lemma 1 $\text{im}(\bar{H}_n(K') \rightarrow \bar{H}_n(L'))$ is finitely generated. Since $\bar{H}_n(K') \rightarrow \bar{H}_n(K)$ is an epimorphism (see [VN or D₃]) we get that $\text{im}(\bar{H}_n(K) \rightarrow \bar{H}_n(L))$ is finitely generated. By Lemma 1, Y is hlc^n .

THEOREM 1. Let X be a locally compact metrizable space such that $X - \{x_0\}$ is hlc^n for some $x_0 \in X$ and $n \geq 0$. If there exists a neighborhood V of x_0 in X such that the image of the inclusion induced homomorphism $\bar{H}_k^c(V) \rightarrow \bar{H}_k^c(X)$ is finitely generated for $k \leq n$, then X is hlc^n .

PROOF. Let U be a compact neighborhood of x_0 in X . Choose a compact neighborhood V of x_0 in U such that $\text{im}(\bar{H}_k^c(V) \rightarrow \bar{H}_k^c(X))$ is finitely generated. Choose a closed subset A of X such that $X = V \cup A$ and $x_0 \notin A$. By excision (see [M, p. 275]) the inclusion induced homomorphism $\bar{H}_m^c(A, V \cap A) \rightarrow \bar{H}_m^c(X, V)$ is an isomorphism for all m . From the diagram

$$\begin{array}{ccc} H_{k+1}^c(A, A \cap V) & \rightarrow & \bar{H}_{k+1}^c(X, V) \\ \partial \downarrow & & \partial \downarrow \\ H_k^c(V \cap A) & \rightarrow & \bar{H}_k^c(V) \\ & & \downarrow \\ & & H_k^c(X) \end{array}$$

we get that the cokernel of $\bar{H}_k^c(V \cap A) \rightarrow \bar{H}_k^c(V)$ is finitely generated for $k \leq n$. Indeed, it is an image of the cokernel of $\partial: \bar{H}_{k+1}^c(X, V) \rightarrow \bar{H}_k^c(V)$ which is isomorphic to the image of $\bar{H}_k^c(V) \rightarrow \bar{H}_k^c(X)$ known to be finitely generated. Choose $a_1, \dots, a_m \in \bar{H}_n^c(V)$ generating the cokernel of $\bar{H}_n^c(V \cap A) \rightarrow \bar{H}_n^c(V)$. Since $X - \{x_0\}$ is hlc^n , the image of $\bar{H}_n^c(V \cap A) \rightarrow \bar{H}_n^c(U)$ is finitely generated (see Lemma 1).

Choose $b_1, \dots, b_p \in \bar{H}_n^c(V \cap A)$ generating the image of that homomorphism. Then $a_1, \dots, a_m, b_1, \dots, b_p$ generate the image of $\bar{H}_n^c(V) \rightarrow \bar{H}_n^c(U)$. By Lemma 1, X is hlc^n .

COROLLARY 3. *If X is a locally compact metrizable space such that $\bar{H}_k^c(X)$ is finitely generated for $k \leq n$ and $X - \{x_0\}$ is hlc^n , then X is hlc^n .*

COROLLARY 4. *Let $X \in \text{hlc}^n$ be a noncompact locally compact metrizable space. The one-point compactification wX of X is hlc^n iff $\bar{H}_k(X)$ is finitely generated for $k \leq n$.*

PROOF. It follows immediately from Theorem 1, Corollary 1 and the fact that the groups $\bar{H}_n^c(wX)$, $\check{H}_n(wX)$, and $\bar{H}_n(X)$ are isomorphic (see [M, p. 86]).

COROLLARY 5. *Let A be a compact metrizable space. If $X \in \text{hlc}^n$ is a compact metrizable space containing A , then the following conditions are equivalent:*

- (a) X/A is hlc^n ,
- (b) $\bar{H}_k(A)$ is finitely generated for $k < n$.

PROOF. X/A is a one-point compactification of $X - A$. Using Corollary 1 and the exact sequence

$$\bar{H}_k(A) \rightarrow \bar{H}_k(X) \rightarrow \bar{H}_k(X - A) \rightarrow \bar{H}_{k-1}(A)$$

(see [M, p. 86]) one establishes the equivalence of conditions (a) and (b).

REMARK. With a little bit more work one can assume in Corollary 5 that X is a locally compact metrizable space.

Corollary 5 and the next result explain the relation between $[D_1]$ and this paper.

LEMMA 2. *For a compact metrizable space X the following conditions are equivalent ($n \geq 0$):*

- (a) $\bar{H}_k(X)$ is finite generated for $k \leq n$,
- (b) $\bar{H}_k(X)$ is countable for $k \leq n$,
- (c) $\text{pro-}H_k(X)$ is stable for $k \leq n$ and satisfies the Mittag-Leffler condition for $k = n + 1$.

If one of the above conditions is satisfied, then the natural homomorphism $\bar{H}_k(X) \rightarrow \check{H}_k(X)$ is an isomorphism for $k \leq n$.

PROOF. Let $X = \text{inv lim}(X_i, p_i^{i+1})$, where X_i is a compact polyhedron for each i . Then we have an exact sequence (see [EH, p. 211])

$$0 \rightarrow \lim^1 H_{k+1}(X_i) \rightarrow \bar{H}_k(X) \rightarrow \text{inv lim } H_k(X_i) \rightarrow 0$$

for each k . Observe that $\text{inv lim } H_k(X_i)$ is simply the Čech k th homology group of X . If condition (c) is satisfied, then $\lim^1 H_k(X_i) = 0$ for $k \leq n + 1$ and $\check{H}_k(X)$ is finitely generated for $k \leq n$. Therefore $\bar{H}_k(X) \rightarrow \check{H}_k(X)$ is an isomorphism and condition (a) follows. If condition (b) is satisfied, then $\lim^1 H_{k+1}(X_i)$ is countable for $k \leq n$ and since each $H_{k+1}(X_i)$ is countable, we have $\lim^1 H_{k+1}(X_i) = 0$ for $k \leq n$ (see [G]). Hence $\text{pro-}H_k(X)$ is Mittag-Leffler (see [G or DS, p. 78]) for $k \leq n + 1$ and $\check{H}_k(X)$ is countable for $k \leq n$. By Corollary 6.1.9 of [DS, p. 81], $\text{pro-}H_k(X)$ is stable for $k \leq n$. Thus condition (c) follows from condition (b), and (a) $\Rightarrow$ (b) is trivial.

LEMMA 3. For a locally compact metrizable space X the following conditions are equivalent ($n \geq 1$):

- (a) X is LC^n ,
- (b) X is hlc^n and $X \in LC^1$.

PROOF. (a) $\Rightarrow$ (b) Given the compact neighborhoods U and V of x in X with $V \subset \text{int } U$ choose a compact neighborhood W of x in V such that $\tilde{H}_k(W) \rightarrow \tilde{H}_k(V)$ is trivial for $k \leq n$ (see [H]). Since $\text{pro-}H_{n+1}(\text{int } U)$ is Mittag-Leffler (see [D₁, Theorem 3.3]), the morphism $\text{pro-}H_{n+1}(V) \rightarrow \text{pro-}H_{n+1}(U)$ factors through a tower of groups which is Mittag-Leffler. Hence $\lim^1(\text{pro-}H_{n+1}(V)) \rightarrow \lim^1(\text{pro-}H_{n+1}(U))$ is trivial and from the diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & \lim^1(\text{pro-}H_{n+1}(W)) & \rightarrow & \tilde{H}_n(U) & \rightarrow & \tilde{H}_n(W) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \lim^1(\text{pro-}H_{n+1}(V)) & \rightarrow & \tilde{H}_n(V) & \rightarrow & \tilde{H}_n(V) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \lim^1(\text{pro-}H_{n+1}(U)) & \rightarrow & \tilde{H}_n(U) & \rightarrow & \tilde{H}_n(U) \rightarrow 0
 \end{array}$$

we get that $\tilde{H}_n(W) \rightarrow \tilde{H}_n(U)$ is trivial.

(b) $\Rightarrow$ (a) Using a similar diagram as above one gets that for each neighborhood U of x in X there exists a neighborhood V of x in U such that $\tilde{H}_k(V) \rightarrow \tilde{H}_k(U)$ is trivial for $k \leq n$. Now, $X \in LC^n$ follows from the main result of [H].

Using the results of this section one easily proves the main result of [D₁]:

THEOREM 2. A compact metrizable space X is an LC^n -divisor ($n \geq 1$) iff it is an LC^1 -divisor and $\text{pro-}H_k(X)$ is stable for $k < n$ and Mittag-Leffler for $k = n$.

As an application of our results we prove the following theorem conjectured by R. Daverman and G. Venema.

THEOREM 3. Let $X \in LC^{n-1}$ be an n -dimensional ($n \geq 0$) compactum. If $\tilde{H}_n(X; \mathbb{Z})$ is finitely generated, then $X \in ANR$.

PROOF. From the proof of Lemma 2 one gets that if $\dim Y \leq n$, then the natural homomorphism $\bar{H}_n(Y) \rightarrow \check{H}_n(Y)$ is an isomorphism (the $\lim^1$ part in the exact sequence vanishes). Hence $\bar{H}_n(X)$ is finitely generated and if $A \subset B$ are two compact subsets of X , then $\bar{H}_{n+1}(B, A) = 0$ and the exact sequence $\bar{H}_{n+1}(B, A) \rightarrow \bar{H}_n(A) \rightarrow \bar{H}_n(B)$ implies that $\bar{H}_n(A) \rightarrow \bar{H}_n(B)$ is a monomorphism. Taking $B = X$ we get, by Lemma 1, $X \in hlc^n$. If $n > 1$, then $X \in LC^n$ (by Lemma 3). Since $\dim X = n$, $X \in ANR$ (see [Hu, p. 168]). If $n = 0$, then X is discrete and $X \in ANR$.

Suppose $n = 1$ and choose a neighborhood V of $x \in X$ such that $\bar{H}_1(V) \rightarrow \bar{H}_1(X)$ is trivial. Then $\bar{H}_1(A) = 0$ for all compact subsets A and V and if $f: S^1 \rightarrow V$ is a map, then $A = f(S^1)$ does not contain any simple closed curve and, therefore, is contractible (see [K, Corollary 7, p. 378]). Thus $X \in LC^1$ and $X \in ANR$ in view of $\dim X = 1$.

REFERENCES

- [B] G. Bredon, *Sheaf theory*, McGraw-Hill, New York, 1967.
- [BM] A. Borel and J. C. Moore, *Homology theory for locally compact spaces*, Michigan Math. J. **7** (1960), 137–159.
- [D₁] J. Dydak, *On LC^n -divisors*, Topology Proc. **3** (1978), 319–333.
- [D₂] ———, *Local n -connectivity of quotient spaces and one-point compactifications*, Shape Theory and Geom. Top. Proc. (Dubrovnik, 1981), Lecture Notes in Math., vol. 870, Springer-Verlag, Berlin and New York, 1981, pp. 48–72.
- [D₃] ———, *An addendum to the Vietoris-Begle theorem*, Topology Appl. (to appear).
- [D₄] ———, *Some properties of nearly 1-movable continua*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **25** (1977), 685–689.
- [DS] J. Dydak and J. Segal, *Shape theory: An introduction*, Lecture Notes in Math., vol. 688, Springer-Verlag, Berlin and New York, 1978.
- [EH] D. A. Edwards and H. M. Hastings, *Čech and Steenrod homotopy theories with applications to geometric topology*, Lecture Notes in Math., vol. 542, Springer-Verlag, Berlin and New York, 1976.
- [G] B. I. Gray, *Spaces of the same n -type, for all n* , Topology **5** (1966), 241–243.
- [H] W. Hurewicz, *Homotopie, Homologie, und lokaler Zusammenhang*, Fund. Math. **25** (1935), 467–485.
- [Hu] S.-T. Hu, *Theory of retracts*, Wayne State Univ. Press, Detroit, Mich., 1965.
- [K] K. Kuratowski, *Topology*, vol. 11, Academic Press, New York, 1968.
- [M] W. S. Massey, *Homology and cohomology theory*, Dekker, New York, 1978.
- [Mil] J. Milnor, *On axiomatic homology theory*, Pacific J. Math. **12** (1962), 337–341.
- [Sh] N. Shrinkhande, *Homotopy properties of decomposition spaces*, Notices Amer. Math. Soc. **22** (1975), Abstract 757-638.
- [VN] A. Jn. Volovikov and Nguen Le Anh, *On the Vietoris-Begle theorem*, Vestnik Moskov. Univ. Ser. I Math. Mekh. (1984), 70–71.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37996