ON THE NUMBER OF GENERATORS AND THE MODULAR GROUP-RING OF A FINITE p -GROUP

OLAF MANZ AND REINER STASZEWSKI

ABSTRACT. We consider the Loewy-series $J(KP)^i$ of a finite p -group P over a field K of characteristic p . We point that the series is not 'monotonic' in general, but we can show that the dimensions of the Loewy-factors $J(KP)^i/J(KP)^{i+1}$ (except the first and the last one) are greater than or equal to the minimal number of generators of P .

Let P be a finite p -group and K an arbitrary field of characteristic p (p a prime). In his paper [3], S. Jennings determined the dimensions of the Loewy-factors $J(KP)^i/J(KP)^{i+1}$ in terms of a certain central series of the group P . In the following, we state his result (cf. [2, Chapter VIII, §§1-2]). Define the central series $P = \kappa_1(P) \geq \kappa_2(P) \geq \dots \geq \kappa_l(P) > \kappa_{l+1}(P) = 1$ by $\kappa_n(P) = [\kappa_{n-1}(P), P]\kappa_m(P)^p$ (for $n > 1$), where m is the least integer for which $pm \geq n$. Then $\kappa_n(P)/\kappa_{n+1}(P)$ is elementary abelian and we put $\dim_{GF(p)}(\kappa_n(P)/\kappa_{n+1}(P)) = d_n$ ($1 \leq n \leq l$). Further, if $J(KP)^s > J(KP)^{s+1} = 0$, we define $\dim_K(J(KP)^i/J(KP)^{i+1}) = c_i$ ($0 \leq i \leq s$).

THEOREM (S. JENNINGS [3]; CF. [2, VIII 2.10]).

$$\prod_{n=1}^l (1 + t^n + \dots + t^{n(p-1)})^{d_n} = \sum_{i=0}^s c_i t^i;$$

in particular, the Loewy-series is symmetric, namely $c_i = c_{s-i}$.

The starting-point for the following paper was a question of B. Huppert: Is the Loewy-series even monotonic, which means $c_{i-1} \leq c_i$ for $1 \leq i \leq s/2$? In the first section, we will show that the answer is 'no'. But we also show that for some classes of p -groups the Loewy-series is indeed monotonic. Let d be the minimal number of generators of P . Then $d = d_1 = c_1$, hence $J(KP)/J(KP)^2 \cong K^d$. In §2, we will prove that the inequality $d \leq c_i$ holds for arbitrary p -groups P ($1 \leq i \leq s-1$).

1. On the monotony of the Loewy-series.

1.1 COUNTEREXAMPLE. As already mentioned, the Loewy-series is not monotonic in general.

(a) Define

$$P = \left\{ \left(\begin{pmatrix} 1 & x & y \\ & 1 & x^3 \\ & & 1 \end{pmatrix} \right) \middle| x, y \in GF(3^3) \right\}.$$

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Then P is a 3-group of order $|P| = 3^6$ and of exponent 3; moreover

$$Z(P) = P' = \left\{ \left(\begin{pmatrix} 1 & 0 & y \\ & 1 & 0 \\ & & 1 \end{pmatrix} \middle| y \in GF(3^3) \right\}.$$

(Indeed, P is the universal 3-group with 3 generators, of class 2 and exponent 3.) With the notation above, we have $\kappa_1(P) = P$, $\kappa_2(P) = P'$, $\kappa_3(P) = 1$, hence $d_1 = d_2 = 3$. Calculating the Jennings-polynomial yields

$$(1 + t + t^2)^3(1 + t^2 + t^4)^3 = \dots + 87t^8 + 85t^9 + 87t^{10} + \dots$$

and the Loewy-series of KP has a waist.

(b) If one tries to check Huppert's question, one would start considering those groups the κ -series of which are determined in [2, VIII 2.9]. For abelian p -groups the conjecture is indeed true (see Proposition 1.4), but we give a counterexample for p -groups of maximal class. Let P be a 3-group of maximal class and order $|P| = 3^5$. Then the Jennings-polynomial is in this case

$$\begin{aligned} (1 + t + t^2)^2(1 + t^2 + t^4)(1 + t^3 + t^6)(1 + t^6 + t^{12}) \\ = \dots + 14t^{12} + 13t^{13} + 14t^{14} + \dots \end{aligned}$$

Now we state some classes of p -groups the Jennings-series of which is monotonic. We call a polynomial $f = \sum_{i=0}^m \alpha_i t^i \in \mathbf{R}[t]$, $\alpha_m \neq 0$, symmetric iff $\alpha_i = \alpha_{m-i}$ ($i = 0, 1, \dots, m$); f is called monotonic iff $\alpha_{i-1} \leq \alpha_i$ ($1 \leq i \leq m/2$).

1.2 LEMMA. *Let $f = \sum_{i=0}^m \alpha_i t^i \in \mathbf{R}[t]$ be symmetric and monotonic. Then so is $f \cdot \left(\sum_{j=0}^n t^j\right)$.*

PROOF. Of course, $\sum_{j=0}^{m+n} \beta_j t^j := f \cdot \left(\sum_{j=0}^n t^j\right)$ is symmetric. We put $\alpha_i = 0$ for negative integers i , hence $\beta_j = \sum_{k=0}^n \alpha_{j-k}$. We have to show that $\beta_j - \beta_{j-1} = \alpha_j - \alpha_{j-1-n}$ is nonnegative for all $0 \leq j \leq (m+n)/2$. This is clear if $j \leq m/2$, because f is monotonic. Suppose therefore that $m/2 < j \leq (m+n)/2$. The symmetry of f shows $\alpha_j = \alpha_{m-j}$, where $m-j \leq m/2$. Now $j \leq (m+n)/2$ yields $j - (n+1) \leq m-j$, which finally means $\alpha_{j-(n+1)} \leq \alpha_{m-j} = \alpha_j$ and $\beta_j - \beta_{j-1} \geq 0$ is shown. $\square$

1.3 LEMMA (HUPPERT, BLACKBURN [2, VIII 2.9]). *Let P be abelian. Then $d_n = 0$ if n is not a power of p and $d_1 \geq d_p \geq d_{p^2} \geq \dots$.*

1.4 PROPOSITION. *If P is abelian, then the Loewy-series of KP is monotonic.*

PROOF. Jennings' Theorem and Lemma 1.3 imply

$$\begin{aligned} \sum_{i=0}^s c_i t^i &= \prod_{k=1}^r (1 + t^{p^k} + \dots + t^{p^k(p-1)})^{d_{p^k}} \\ &= (1 + t + t^2 + \dots + t^{p^{r+1}-1})^{d_{p^r}} \\ &\quad \cdot (1 + t + t^2 + \dots + t^{p^r-1})^{d_{p^{r-1}} - d_{p^r}} \dots \\ &\quad \cdot (1 + t + t^2 + \dots + t^{p^2-1})^{d_p - d_{p^2}} \\ &\quad \cdot (1 + t + t^2 + \dots + t^{p-1})^{d_1 - d_p}. \end{aligned}$$

Now Lemma 1.2 at once yields the result by induction. $\square$

We prove Proposition 1.4 for a larger class of p -groups. Therefore, we need a lemma, analogous to 1.3.

1.5 LEMMA. *Let P be a regular p -group (in the sense of P. Hall) and let $P' \leq P^p$. Then we have $\kappa_n(P) = P^{p^i}$ for $p^{i-1} < n \leq p^i$.*

PROOF. Induction on n . By definition, $\kappa_1(P) = P$. Let $n > 1$, then $\kappa_n(P) = [\kappa_{n-1}(P), P]\kappa_m(P)^p$, where m denotes the minimal natural number with $pm \geq n$. Choose $i > 0$, such that $p^{i-1} < n \leq p^i$; hence $p^{i-2} < m \leq p^{i-1}$.

Case 1. $p^{i-1} + 1 < n \leq p^i$. This yields by induction

$$\kappa_n(P) = [P^{p^i}, P](P^{p^{i-1}})^p = P^{p^i}.$$

Case 2. $n = p^{i-1} + 1$. Hence

$$\begin{aligned} \kappa_{p^{i-1}+1}(P) &= [\kappa_{p^{i-1}}(P), P]\kappa_{p^{i-2}+1}(P)^p \\ &= [P^{p^{i-1}}, P](P^{p^{i-1}})^p \quad (\text{by induction}) \\ &= (P')^{p^{i-1}} P^{p^i} \quad (\text{by [1, III 10.8]}) \\ &= P^{p^i} \quad (\text{as } P' \leq P^p). \quad \square \end{aligned}$$

1.6 PROPOSITION. *Let P be a regular p -group and $P' \leq P^p$. Then the Loewy-series of KP is monotonic.*

PROOF. By Lemma 1.5, $d_n = 0$ if n is not a power of p . On the other hand, $d_1 \geq d_p \geq d_{p^2} \geq \dots$ (by [1, III 10.7]), and the proof of Proposition 1.4 runs through. $\square$

We close §1 considering another class of frequently appearing p -groups.

1.7 PROPOSITION. *If P is an extra-special p -group, then the Loewy-series is monotonic.*

PROOF. Let $|P| = p^{2m+1}$. Clearly, $\kappa_2(P) = Z(P) = \phi(P)$.

Case 1. $\text{Exp}(P) = p^2$. By definition of the κ -series, we obtain $d_1 = 2m$, $d_2 = \dots = d_{p-1} = 0$, $d_p = 1$, $d_i = 0$ ($i > p$), and the proof of Proposition 1.4 yields the result.

Case 2. $\text{Exp}(P) = p$. Then $d_1 = 2m$, $d_2 = 1$ and $d_i = 0$ ($i > 2$). By Lemma 1.2 it is obviously sufficient to consider $|P| = p^3$. Now the Jennings-polynomial is

$$\sum_{i=0}^s c_i t^i = (1 + t + \dots + t^{p-1})^2 (1 + t^2 + \dots + t^{2(p-1)}).$$

If $\sum_{i=0}^{s/2} \alpha_i t^i = (1 + t + \dots + t^{p-1})^2$, then $\alpha_i = i + 1$ for $0 \leq i \leq p - 1$ and $\alpha_{2(p-1)-i} = \alpha_i$. We put again $\alpha_i = 0$ for negative integers i . As $c_j = \sum_{i=0}^{p-1} \alpha_{j-2i}$, we get $c_j - c_{j-1} = \sum_{i=0}^{p-1} (\alpha_{j-2i} - \alpha_{j-1-2i})$. If $j \leq p - 1$, then clearly $c_j - c_{j-1} \geq 0$ and we are done. Suppose therefore $p - 1 < j \leq 2(p - 1)$. We count the number of summands $\alpha_{j-2i} - \alpha_{j-1-2i}$ which are 1 (denoted by n_1) and which are -1 (denoted by n_{-1}).

(a) We have $\alpha_{j-2i} - \alpha_{j-1-2i} = 1$ iff $0 \leq j - 2i \leq p - 1$, hence $n_1 \geq (p - 1)/2$.

(b) On the other hand, $\alpha_{j-2i} - \alpha_{j-1-2i} = -1$ iff $p - 1 < j - 2i \leq 2(p - 1)$, hence $n_{-1} \leq (j - p)/2 + 1$.

By (a) and (b),

$$\begin{aligned} c_j - c_{j-1} &= n_1 - n_{-1} \geq (p-1)/2 - ((j-p)/2 + 1) \\ &\geq (p-1)/2 - ((2(p-1) - p)/2 + 1) \geq -1/2. \end{aligned}$$

As $c_j - c_{j-1}$ is an integer, we have $c_j - c_{j-1} \geq 0$ and the proof is complete. $\square$

2. A lower bound for the orders of the Loewy-factors. Clearly,

$$KP/J(KP) \cong J(KP)^s \cong K.$$

Furthermore, $J(KP)/J(KP)^2 \cong K^d$, where d denotes the minimal number of generators of P . In this section, we show that d is a lower bound for the dimensions of the Loewy-factors $J(KP)^i/J(KP)^{i+1}$ for $1 \leq i \leq s-1$. This generalizes a result of B. Külshammer [4], who proved the following: If $J(KP)^i/J(KP)^{i+1} \cong K$ for some i ($1 \leq i \leq s-1$), then P is cyclic and all Loewy-factors are isomorphic to K .

We start stating some consequences of the definition of the central series $\kappa_n(P)$.

2.1 LEMMA. *If $d_{n+1} = d_{n+2} = \dots = d_{np} = 0$, then $d_i = 0$ for all $i > np$.*

PROOF. Suppose not. We choose a minimal index i such that $\kappa_i(P) > \kappa_{i+1}(P)$ and $i > np$. Let further m be the minimal natural number such that $pm \geq i+1$. By the definition of the κ -series, we have $\kappa_{i+1}(P) = [\kappa_i(P), P]\kappa_m(P)^p$.

We claim that $n+1 \leq m \leq i$: Since $mp \geq i+1 > np$, we conclude $m \geq n+1$. On the other hand, the minimal choice of m yields $i+1 > p(m-1)$, hence $m \leq i$. As $\kappa_i(P) = \kappa_{np}(P)$, the equation $\kappa_{i+1}(P) = [\kappa_{np}(P), P]\kappa_{n+1}(P)^p$ holds. But $n+1$ is the minimal natural number such that $p(n+1) \geq np+1$, hence $\kappa_{i+1}(P) = \kappa_{np+1}(P) = \kappa_i(P)$, a contradiction. $\square$

To simplify our notation, we denote those d_i which are different from 0 by d_{m_1} ($= d_1 = d$), $d_{m_2}, \dots, d_{m_k}, \dots, d_{m_e}$. In these terms, the assertion of Lemma 2.1 runs as follows: $m_k \leq pm_{k-1}$ ($2 \leq k \leq e$). In addition, we define $s_k = (p-1) \sum_{i=1}^k m_i d_{m_i}$; by Jennings' Theorem $s_e = s$.

2.2 LEMMA. *With the notations above, we have*

$$s_k \geq pm_k + (p-1)(d-1) - 1.$$

PROOF. We use induction on k .

$$\begin{aligned} k=1: \quad s_1 &= (p-1)m_1 d_{m_1} = pm_1 d - d = pm_1 + p(d-1) - d \\ &= pm_1 + (p-1)(d-1) - 1. \end{aligned}$$

$$\begin{aligned} k > 1: \quad s_k &= s_{k-1} + (p-1)m_k d_{m_k} \\ &\geq pm_{k-1} + (p-1)(d-1) - 1 + (p-1)m_k d_{m_k} \quad (\text{by induction}) \\ &\geq pm_k + (p-1)(d-1) - 1 + (pm_{k-1} - m_k) \quad (\text{as } d_{m_k} \geq 1) \\ &\geq pm_k + (p-1)(d-1) - 1 \quad (\text{by Lemma 2.1}). \quad \square \end{aligned}$$

As an easy consequence of Lemmata 2.1 and 2.2 we obtain

2.3 LEMMA. $m_k \leq s_{k-1} - (p-1)(d-1) + 1$.

We are now able to prove the main result of this section.

2.4 THEOREM. Let d as before be the minimal number of generators of the p -group P and c_n ($0 \leq n \leq s$) the dimensions of $J(KP)^n/J(KP)^{n+1}$, the Loewy-factors. Then

$$c_n \geq d \quad (1 \leq n \leq s-1).$$

PROOF. If $d = 1$, then P is cyclic and all $c_n = 1$. Therefore, we may assume $d \geq 2$. We put

$$\begin{aligned} f_k &= \prod_{j=1}^k (1 + t^{m_j} + \dots + t^{(p-1)m_j})^{d_{m_j}} \\ &=: \sum_{n=0}^{s_k} c_n^{(k)} t^n \quad \text{for } k = 1, \dots, e. \end{aligned}$$

In particular, $c_n^{(e)} = c_n$. Of course, $c_1^{(k)} = d_{m_1} = d$ for all k . We show by induction on k : $c_n^{(k)} \geq d$ for $1 \leq n \leq s_k - 1$.

$$k = 1: \quad f_1 = (1 + t^{m_1} + \dots + t^{(p-1)m_1})^{d_{m_1}} = (1 + t + \dots + t^{(p-1)})^d.$$

By Lemma 1.1, f_1 is symmetric and monotonic. As $c_1^{(1)} = d$, the assertion holds.

$$k > 1: \quad f_k = f_{k-1} (1 + t^{m_k} + \dots + t^{(p-1)m_k})^{d_{m_k}}$$

and the induction yields $c_n^{(k-1)} \geq d$ for $1 \leq n \leq s_{k-1} - 1$.

Let $1 \leq l \leq s_k - 1$. Then

$$c_l^{(k)} \geq \sum_{i=0}^{(p-1)d_{m_k}} c_{l-im_k}^{(k-1)},$$

where the $c_r^{(k-1)}$ are understood to be 0 for negative indices r . Clearly, $1 - (p-1)d_{m_k}m_k \leq s_{k-1} - 1$, as $s_k - 1 - (p-1)d_{m_k}m_k = s_{k-1} - 1$. Moreover, Lemma 2.3 yields $m_k \leq s_{k-1}$.

(1) Suppose at first $m_k \leq s_{k-1} - 1$. Then $M := \{l, l - m_k, l - 2m_k, \dots, l - [(p-1)d_{m_k}]m_k\} \cap \{1, 2, \dots, s_{k-1} - 1\} \neq \emptyset$. We take $n \in M$ and our induction hypothesis shows $c_l^{(k)} \geq c_n^{(k-1)} \geq d$.

(2) It remains to consider the case $m_k = s_{k-1}$. If even in this case $M \neq \emptyset$, we can proceed as in (1). Finally, let $M = \emptyset$; then there is an index i such that $l - im_k = 0$ and $l - (i-1)m_k = s_{k-1}$, consequently $c_l^{(k)} \geq c_0^{(k-1)} + c_{s_{k-1}}^{(k-1)} = 2$. On the other hand, the assumption (2) and Lemma 2.3 show $p = d = 2$, hence $c_l^{(k)} \geq 2 = d$. $\square$

As already mentioned, the Loewy-series remains stationary from the beginning (which means of course all Loewy-factors are isomorphic to K) if and only if P is cyclic. In view of Theorem 2.4, it seems to be of interest to determine those groups P for which the Loewy-series is stationary from the second up to the penultimate factor.

2.5 PROPOSITION. The Loewy-series of KP is stationary from the second up to the penultimate factor (which means $d = c_1 = c_2 = \dots = c_{s-1}$) if and only if one of the following cases occurs:

- (1) P is cyclic,
- (2) $p = 2$ and $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,
- (3) $p = 2$ and P has a cyclic normal subgroup of index 2.

PROOF. “ $\Leftarrow$ ” If P is cyclic, then all Loewy-factors are isomorphic to K . If $P \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, then the Jennings-polynomial is $(1+t)^3 = 1 + 3t + 3t^2 + t^3$.

So, let P have a cyclic normal subgroup C of index 2 and $p = 2$. It is easy to see that $\kappa_j(P) = C^{2^i}$ for $2^{i-1} < j \leq 2^i$ and $j \geq 2$. Hence we obtain the Jennings-polynomial of P (for some integer n)

$$\begin{aligned} & (1+t)^2(1+t^2)(1+t^4) \cdots (1+t^{2^n}) \\ &= (1+t) \cdot \sum_{i=0}^{2^{n+1}-1} t^i = 1 + 2 \sum_{i=1}^{2^{n+1}-1} t^i + t^{2^{n+1}}. \end{aligned}$$

Therefore, all but the first and the last Loewy-factors have dimension 2.

“ $\Rightarrow$ ” If $s = 1$ or $s = 2$, then $P \cong \mathbf{Z}_2, \mathbf{Z}_3$ or $\mathbf{Z}_2 \times \mathbf{Z}_2$. Hence, we may assume $s \geq 3$. To calculate c_1 and c_2 it is obviously enough to consider the polynomial

$$(1+t+t^2+\cdots+t^{p-1})^d \cdot (1+t^2+\cdots+t^{2(p-1)})^{d_2}.$$

(Note that d_2 may be 0.) Of course, we have $c_1 = d$.

(a) Suppose first $p > 2$. Then $c_2 = d + \binom{d}{2} + d_2$ and $c_2 = c_1$ yields $d_2 = 0$ and $d = 1$, hence P is cyclic.

(b) In the case $p = 2$ we obtain $c_2 = \binom{d}{2} + d_2$ and $c_2 = c_1$ yields $d^2 - 3d + 2d_2 = 0$, consequently $d \leq 3$.

(1) If $d = 1$, then P is cyclic.

(2) Secondly, $d = 3$ forces $d_2 = 0$. This implies $d_i = 0$ for all $i \geq 2$ (by Lemma 2.1), hence P is elementary abelian and $P \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

(3) Finally, suppose $d = 2$, hence $d_2 = 1$ and $2 = d = c_1 = \cdots = c_{s-1}$. We put as in the proof of Theorem 2.4

$$f_k = \prod_{j=1}^k (1+t^{m_j})^{d_{m_j}} = \sum_{n=0}^{s_k} c_n^{(k)} t^n \quad (k = 1, \dots, e).$$

Trivially, $c_1^{(k)} = \cdots = c_{s_k-1}^{(k)} = 2$ for all k .

(a) We claim $d_{m_k} = 1$ for all $k \geq 2$. Suppose not. Hence

$$\begin{aligned} f_k &= f_{k-1} \cdot (1+t^{m_k})^{d_{m_k}} = (1+2t+\cdots+2t^{s_{k-1}-1}+t^{s_{k-1}}) \\ &\quad \cdot (1+d_{m_k}t^{m_k}+\cdots+t^{d_{m_k}m_k}) \end{aligned}$$

and the coefficient of t^{m_k+1} is at least $2 \cdot d_{m_k} > 2$, a contradiction.

(β) We now claim that $P/\kappa_{m_k+1}(P)$ has a cyclic normal subgroup of index 2 ($k = 1, \dots, e$): We use induction on k . If $k = 1$, then $P/\kappa_2(P)$ is of type $(2, 2)$. In case of $k = 2$, we have $m_2 = 2$ (as $d_2 = 1$) and $P/\kappa_3(P)$ has order 8 and 2 generators. Let $k > 2$. By (α), $\kappa_{m_k}(P)/\kappa_{m_k+1}(P)$ has order 2. Let $Z = \kappa_{m_k}(P) = \kappa_{m_{k-1}+1}(P)$ and write $\bar{P} = P/\kappa_{m_k+1}(P)$. By the induction hypothesis, $\bar{P}/\bar{Z}$ has a normal cyclic subgroup $\langle x\bar{Z} \rangle$ of index 2. If $\bar{Z} = \langle z \rangle$ and z is a power of x , certainly $\langle x \rangle$ is the desired subgroup of $\bar{P}$. In the other case, $\langle x \rangle \times \langle z \rangle \cong \mathbf{Z}_{2^a} \times \mathbf{Z}_2$ has index 2 in $\bar{P}$. But now it is clear that $z \notin \kappa_3(\bar{P})$, a contradiction. As $\kappa_{m_e+1}(P) = 1$, (β) completes the proof. $\square$

2.6 REMARK. The 2-groups with a cyclic normal subgroup of index 2 are completely described in Huppert [1, I, 14.9]. The list consists of abelian groups,

the quaternion, dihedral and semidihedral groups; furthermore the groups $P = \langle g, h/g^{2^n} = h^2 = 1, g^h = g^{1+2^{n-1}} \rangle$ belong to the list.

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FACHBEREICH MATHEMATIK, UNIVERSITÄT MAINZ, SAARSTRASSE 21, 6500 MAINZ,
WEST GERMANY