NOTE ON SYMBOLIC POWERS AND GOING DOWN

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ABSTRACT. For primes $P \subseteq Q$ of a Noetherian ring R , we consider when, for all $k \geq 1$, there is an m with $P^{(m)} \subseteq Q^{(k)}$, and reprove a relevant theorem of Schenzel. If R is a domain, we consider sufficient conditions for $P \subseteq Q$ to satisfy going down for all primes Q containing P .

Introduction. Let P be a prime ideal in a Noetherian ring R . In [7], Schenzel shows that $E(P)$, the set of essential prime divisors of P (defined below), equals $\{P\}$ if and only if the P -adic and P -symbolic topologies are equivalent. In the first section, we offer a new, and fairly easy proof. Our arguments shed light on when, for primes $P \subseteq Q$, for all $k \geq 1$, there is an m with $P^{(m)} \subseteq Q^{(k)}$. The second section considers a prime P in a Noetherian domain R , and shows that if $E(P) = \{P\}$, then $P \subseteq Q$ satisfies going down for all primes Q containing P . In §3, we list some circumstances which assure that $E(P) = \{P\}$.

1. Symbolic powers.

Notation. R will always be a Noetherian ring. Its integral closure will be denoted R' , and if it is local, its completion will be R^* . If I and J are ideals of R , then the eventual stable value of $(I: J) \subseteq (I: J^2) \subseteq \dots$ will be denoted by $I: \langle J \rangle$. For I an ideal in R , $A^*(I) = \text{Ass } R/I^n$ for all large n , while $\bar{A}^*(I) = \text{Ass } R/(I^n)_a$ for all large n , $(I^n)_a$ denoting the integral closure of I^n . Finally, $E(I) = \{P \in \text{Spec } R \mid I \subseteq P \text{ and there is a } z \in \text{Ass } R_P^* \text{ with } P_P^* \text{ minimal over } IR_P^* + z\}$.

$A^*(I)$ and $\bar{A}^*(I)$ are discussed in [2, Chapters 1 and 3], where it is shown that both are well defined finite sets. $E(I)$ is discussed in [4]. In our first lemma we list the facts about $E(I)$ which we need.

LEMMA 1.1. *Let I be an ideal in R .*

- (i) $E(I) \subseteq A^*(I)$.
- (ii) *If Q is a prime containing I , then $Q \in E(I)$ if and only if there is an integer $k \geq 1$ such that, for all $m \geq 1$, $I^m: \langle Q \rangle \not\subseteq Q^{(k)}$.*
- (iii) *If the prime P is minimal over $I + z$, with $z \in \text{Ass } R$, then $P \in E(I)$.*
- (iv) *If $P \in \text{Spec } R$ and S is a multiplicatively closed subset of R disjoint from P , then $P \in E(I)$ if and only if $P_S \in E(I_S)$.*
- (v) *If $R \subseteq T$ is a finite integral extension of Noetherian domains, then $P \in E(I)$ if and only if there is a $Q \in E(IT)$ with $Q \cap R = P$.*

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(vi) Let T be a faithfully flat Noetherian extension ring of R . If $Q \in E(IT)$ then $Q \cap R \in E(I)$. If $P \in E(I)$ and if Q is a prime minimal over PT , then $Q \cap R = P$ and $Q \in E(IT)$.

(vii) Let $X_1, \dots, X_n$ be indeterminates. Then

$$E(IR[X_1, \dots, X_n]) = \{PR[X_1, \dots, X_n] \mid P \in E(I)\}.$$

PROOF. (i) through (vi) follows from [4, 3.3.1, 3.2, 3.3.4, 3.3.2, 3.9, and 3.7]. For (vii), one containment follows from (vi). For the other, assume that $Q \in E(IR[X_1, \dots, X_n])$, and that $Q \cap R = P$. By (vi), $P \in E(I)$. By (i), Q is a prime divisor of $I^m R[X_1, \dots, X_n]$, for large m . It is well known [5, 18.11] that this implies $Q = PR[X_1, \dots, X_n]$, proving (vii).

We come to our main result.

THEOREM 1.2. *Let $P \subset Q$ be primes in R . Concerning the following three statements, (a) $\Rightarrow$ (b) $\Rightarrow$ (c).*

(a) $Q \in E(P)$.

(b) For some $k \geq 1$, $P^{(m)} \not\subseteq Q^{(k)}$ for all $m \geq 1$.

(c) There is a prime q with $P \subset q \subseteq Q$, and $q \in E(P)$.

PROOF. (a) $\Rightarrow$ (b). By Lemma 1.1(ii), we have $P^m : \langle Q \rangle \not\subseteq Q^{(k)}$ for some $k \geq 1$, and all $m \geq 1$. However, $P^m : \langle Q \rangle \subseteq P^{(m)} : \langle Q \rangle \subseteq P^{(m)}$, the last inclusion since $P^{(m)}$ is P -primary, and Q properly contains P . It follows that (b) holds.

(b) $\Rightarrow$ (c). We will prove the contrapositive. For this, we may localize at Q . Thus we will assume that (R, Q) is local, and that $E(P) = \{P\}$. For any $k \geq 1$, we seek an m with $P^{(m)} \subset Q^{(k)} = Q^k$. Let $S = R - P$. If $z \in \text{Ass } R^*$, let q_z be a prime minimal over $PR^* + z$. By Lemma 1.1(iii) and (vi), $q_z \cap R \in E(P) = \{P\}$. Thus $q_z \cap S = \emptyset$ and $z \cap S = \emptyset$. The first of these shows that $PR_S^* + z_S \subseteq (q_z)_S \neq R_S^*$ for all $z \in \text{Ass } R^*$. The Krull Intersection Theorem shows that $\bigcap \{P^n R_S^* \mid n \geq 1\} = 0$. However, since $z \cap S = \emptyset$ for all $z \in \text{Ass } R^*$, S consists of regular elements of R^* , and R^* is embedded in R_S^* . Therefore $\bigcap \{P^n R_S^* \cap R^* \mid n \geq 1\} = 0$. Because R^* is complete, for $k \geq 1$, there is an m with $P^m R_S^* \cap R^* \subseteq Q^{*k}$ [5, 30.1]. Obviously, $P^{(m)} \subseteq P^m R_S^* \cap R$, and of course $Q^{*k} \cap R = Q^k$. Therefore, $P^{(m)} \subseteq Q^k$, as desired.

COROLLARY 1.3. *Let $P \subset Q$ be primes in R , and suppose that no prime properly between P and Q is an $A^*(P)$. Then $Q \notin E(P)$ if and only if for all $k \geq 1$ there is an m such that $P^{(m)} \subseteq Q^{(k)}$. In particular, this equivalence holds whenever $\text{height } Q/P = 1$.*

PROOF. If $Q \notin E(P)$, then the hypothesis and Lemma 1.1(i) show that statement (c) of Theorem 1.2 is false. Thus (b) is false, giving one direction of our equivalence. The other direction is by the contrapositive of (a) $\Rightarrow$ (b) of that theorem. The second statement is immediate from the first.

For comparison to Corollary 1.3, we present the next result. (The converse of this fails.)

PROPOSITION 1.4. *Let $P \subset Q$ be primes in R , and suppose that no prime properly between P and Q is in $A^*(P)$. If $Q \notin A^*(P)$, then for all large k , $P^{(k)} \subset Q^{(k)}$. (This holds for all but finitely many primes Q such that $\text{height } Q/P = 1$.)*

PROOF. Suppose $Q \notin A^*(P)$. Let k be large enough that $A^*(P) = \text{Ass } R/P^k$. The hypothesis shows that P_Q is the only prime divisor of P_Q^k , so that $P_Q^{(k)} = P_Q^k$.

As $P_Q^k \subset Q_Q^k$, we get $P^{(k)} \subset Q^{(k)}$. (The last statement is by the finiteness of $A^*(P)$.)

In [7], Schenzel shows that $E(P) = \{P\}$ if and only if the P -adic and P -symbolic topologies are equivalent. (That is, if and only if for all $k \geq 1$ there is an m with $P^{(m)} \subseteq P^k$.) We get this as a corollary to Theorem 1.2.

COROLLARY 1.5. *Let P be a prime in R . Then $E(P) = \{P\}$ if and only if the P -adic and P -symbolic topologies are equivalent.*

PROOF. Suppose $E(P) = \{P\}$. By Theorem 1.2(b) $\Rightarrow$ (c), if $P \subseteq Q \in \text{Spec } R$, then for all $h \geq 1$, there is an n with $P^{(n)} \subseteq Q^{(h)}$. Applying this to the primes Q involved in a primary decomposition of P^k (any $k \geq 1$), it is easy to find an m with $P^{(m)} \subseteq P^k$. The converse is easy, using Theorem 1.2(a) $\Rightarrow$ (b), and the fact that $P^k \subset Q^k \subseteq Q^{(k)}$, for any prime $Q \supset P$.

REMARK. It is easy to see that the following three statements are equivalent for a prime P in R . In this paper, we will only refer to the first of them. (i) The P -adic and P -symbolic topologies are equivalent. (ii) If $P \subseteq Q \in \text{Spec } R$, and if $k \geq 1$, then there is an m with $P^{(n)} \subseteq Q^k$. (iii) If $P \subseteq Q \in \text{Spec } R$, and if $k \geq 1$, then there is an m with $P^{(m)} \subseteq Q^{(k)}$.

REMARK. The P -adic and P -symbolic topologies are called linearly equivalent if there is an $h \geq 0$ such that $P^{(n)} \subseteq P^{n-h}$ for all $n \geq h$. This was first studied in [7]. In [6] (and [3]) it is shown to occur exactly when $U(P) = \{P\}$. Here, $U(I) = \{Q \cap R \mid Q \in E(t^{-1}\mathbf{R}(I))\}$, with $\mathbf{R}(I) = R[t^{-1}, It]$ the Rees ring of R with respect to I .

2. Going down. Recall that primes $P \subseteq Q$ in a domain R satisfy going down if for any integral extension domain T of R , and $q \in \text{Spec } T$ with $q \cap R = Q$, there is a $p \in \text{Spec } T$ with $p \subseteq q$ and $p \cap R = P$. [2, Corollary 9.2] shows that if P is prime in a Noetherian domain, and if $\overline{A}^*(P) = \{P\}$, then $P \subseteq Q$ satisfies going down for all primes Q containing P . We give a new proof of this which also shows that the result remains true if $\overline{A}^*(P) = \{P\}$ is replaced by $E(P) = \{P\}$. (In the next section, we will discuss hypotheses under which $E(P) = \{P\}$, so that in those situations, going down will automatically hold.) In this section, all rings will be domains.

DEFINITION. Let I be an ideal in the domain R . $M(I) = \{q \cap R \mid q \text{ is a prime minimal over } IT, \text{ for some integral extension domain } T \text{ of } R\}$.

Our next easy lemma follows directly from [1, exercise 37(i) $\Leftrightarrow$ (ii), §1.6]. (It does not need R to be Noetherian.)

LEMMA 2.1. *If $P \in \text{Spec } R$, with R a domain, then $P \subseteq Q$ satisfies going down for all primes Q which contain P if and only if $M(P) = \{P\}$.*

LEMMA 2.2. *If R is a Noetherian domain, then in the definition of $M(I)$, we may assume that $T = R'$. Alternately, we may assume that $T = R[x]$ for some $x \in R'$.*

PROOF. Starting with q and T as in the definition, by lying over and incomparability, we may assume that $T = T'$. Since $R' \subseteq T'$ satisfies going down, we may assume that $T = R'$. Since only finitely many primes of R' lie over $q \cap R$ [5, 33.10], pick x in q but in no other prime of R' lying over $q \cap R$. Then q is the only prime

of R' lying over $q \cap R[x]$, and so by lying over and going up, we may assume that $T = R[x]$.

The next lemma, while easy, is our key observation in this section.

LEMMA 2.3. *Let I be an ideal in a Noetherian domain. Then $M(I) \subseteq E(I) \cap A^*(I) \cap \bar{A}^*(I)$.*

PROOF. Say $Q \in M(I)$. By Lemma 2.2, there is a finite integral extension domain T of R , and q minimal over IT with $q \cap R = Q$. The definitions of $E(IT)$ and $\bar{A}^*(IT)$ make it clear that they contain all primes minimal over IT . Thus $q \in E(IT) \cap \bar{A}^*(IT)$. The result follows from Lemma 1(v) and (i), and [2, Proposition 3.22].

COROLLARY 2.4. *Let P be a prime in a Noetherian domain. If either $A^*(P) = \{P\}$, or $\bar{A}^*(P) = \{P\}$, or $E(P) = \{P\}$, then $P \subseteq Q$ satisfies going down for all primes Q containing P .*

PROOF. This is immediate from Lemmas 2.1 and 2.3.

3. $E(P) = \{P\}$.

LEMMA 3.1. *If (R, M) is such that the completion R^* contains a single prime divisor of zero, z , and if I is an ideal of R , then $M \in E(I)$ if and only if M is minimal over I .*

PROOF. Suppose $I \subseteq P \subset M$, with P prime. By [5, 18.11] there is a prime p in R^* lying over P . Thus $IR^* \subseteq p$. Also, by its uniqueness, $z \subseteq p$. Thus M^* is not minimal over $IR^* + z$, and so $M \notin E(I)$. The converse follows from the definition of $E(I)$.

Recall that the local domain (R, M) is analytically irreducible if its completion is a domain (so in this case, Lemma 3.1 applies). Also, (R, M) is analytically normal means that the completion is a normal domain.

LEMMA 3.2. *Let I be an ideal in the Noetherian domain R . Suppose there is a finite integral extension domain A of R such that for all $Q \in \text{Spec } A$ with $I \subset Q \cap R$, A_Q^* has a single prime divisor of zero. Then $M(I) = E(I) = \{Q \cap R \mid Q \text{ is minimal over } IA\}$. In particular, if R is a domain finitely generated over a field, then for any ideal I , $M(I) = E(I) = \{Q \cap R \mid Q \text{ is minimal over } IR'\}$.*

PROOF. Clearly, $\{Q \cap R \mid Q \text{ is minimal over } IA\} \subseteq M(I) \subseteq E(I)$ (by Lemma 2.3). Suppose $P \in E(I)$. If $P = I$, then clearly P is in the first set. Thus, suppose that $I \subset P$. By Lemma 1.1(v), there is a $Q \in E(IA)$ with $Q \cap R = P$. By Lemma 3.1, Q is minimal over IA . This shows our three sets are equal. As for the final statement, for a domain R finitely generated over a field, it is well known that R'_Q is analytically irreducible for all $Q \in \text{Spec } R'$, so that R' satisfies the hypothesis on A .

PROPOSITION 3.3. *Let P be a prime in a Noetherian domain R , and consider the following four statements.*

- (a) $E(P) = \{P\}$.
- (b) The P -adic and P -symbolic topologies are equivalent.

(c) any prime minimal over PR' lies over P .

(d) For all primes Q containing P , $P \subseteq Q$ satisfies going down.

Then

(i) (a) $\Leftrightarrow$ (b), (c) $\Leftrightarrow$ (d), and (a) $\Rightarrow$ (d).

(ii) If R is finitely generated over a field, then these statements are all equivalent.

(iii) If R is finitely generated over a field, if $\text{height } P = h$, and if P is the radical of an ideal generated by h elements, then these statements are all true.

(iv) If R is locally analytically irreducible, then these statements are all true.

(v) If for every maximal ideal M of R , R_M is analytically irreducible, then these statements are all true.

(vi) If R is a finite integral extension domain of $L[X_1, \dots, X_n]$, with L an analytically irreducible local domain and $X_1, \dots, X_n$ indeterminates, and if P is the radical of pR for a depth 1 prime p of L , then these statements are all true.

PROOF. (i) Corollary 1.5 gives (a) $\Leftrightarrow$ (b). Lemmas 2.1 and 2.2 give (c) $\Leftrightarrow$ (d). Corollary 2.4 gives (a) $\Rightarrow$ (d).

(ii) Lemma 3.2 gives (a) $\Leftrightarrow$ (c). The rest follows from (i).

(iii) Suppose that $\text{height } P = h$ and P is the radical of $(x_1, \dots, x_h)R$. We will show that (c) holds for P . Let Q be minimal over PR' . (Note that R' is a finite R -module [5, 36.6].) Since P is the radical of $(x_1, \dots, x_h)R$, it is easily seen that Q is also minimal over $(x_1, \dots, x_h)R'$. From this, it follows that $\text{height } Q \leq h$. Now [5, 34.8] shows

$$\text{height } Q \cap R = \text{height } Q \leq h.$$

However, $P \subseteq Q \cap R$, and $\text{height } P = h$, so $P = Q \cap R$. Thus (c) holds. By (ii), all of the statements hold.

(iv) By (i), it will suffice to show that (a) holds. However, Lemma 3.1 shows that no prime properly containing P can be in $E(P)$, and so (a) does hold.

(v) By (iv), it will suffice to show that R is locally analytically irreducible. Let Q be a prime in R , and let M be a maximal prime containing Q . Let q be a prime minimal over QR_M^* . Since $(R_M^*)/p$ is complete for every prime p in R_M^* , [5, 32.1] shows that R_M^* is pseudo-geometric, as is $(R_M^*)_q$, using [5, 36.1]. As $(R_M^*)_q$ is of finite type over R_M^* , [5, 37.8] shows that $(R_M^*)_q$ is analytically irreducible. Finally, by [5, 19.2(3)], R_Q is a subspace of $(R_M^*)_q$, and so R_Q is analytically irreducible. Thus R is locally analytically irreducible.

(v) Since L is analytically irreducible and $\text{depth } p = 1$, Lemma 3.1 shows that $E(p) = \{p\}$. Let $q = pL[X_1, \dots, X_n]$. By Lemma 1(vii), $E(q) = \{q\}$. Since $\text{Rad } pR = P$, Lemma 1.1(v) and incomparability show $E(pR) = \{P\}$. Finally, since $\text{Rad } pR = P$, the definition easily shows that $E(P) = E(pR) = \{P\}$. Thus (a) holds. By (i), all the statements hold.

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