ON BINOMIAL UNITS OF CERTAIN CUBIC FIELD

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ABSTRACT. Scarowsky has conjectured about a binomial unit in a cubic field. We discuss a relation between binomial units and a diophantine equation.

In a recent paper, Scarowsky [3] has conjectured about a binomial unit in a cubic field. We prove the following

THEOREM. *Let $K = \mathbb{Q}(\theta)$ be a cubic field, where $\theta^3 + 12a\theta - 12 = 0$, $a > 0$. If $\varepsilon = 1 - a\theta$ is the fundamental unit of K , it follows that all solutions of $x^3 + 12axy^2 - 12y^3 = 1$ are $(x, y) = (1, 0), (1, a)$.*

LEMMA 1 [1]. *If $b\theta + c$, where $b \neq 0, \pm 1$, is a positive unit in $\mathbb{Z}[\theta]$ where $\theta^3 - P\theta^2 + Q\theta - R = 0$, then no power > 1 of $b\theta + c$ can be a binomial unit.*

LEMMA 2 [2]. *Let $\varepsilon = \pm\theta + c$ be a unit in $\mathbb{Z}[\theta]$, where $\theta^3 - P\theta^2 + Q\theta - R = 0$. If $\theta^3 \equiv 0 \pmod{p^2}$, where p is a prime, then $p \nmid c$ and $\varepsilon^n = u + v\theta$ is impossible for $n > 1$.*

The Diophantine equation $x^3 + 12axy^2 - 12y^3 = 1$ can be written as a norm equation

$$N_{K/\mathbb{Q}}(x - \theta y) = 1.$$

All solutions of the Diophantine equation are given by $x - \theta y = (1 - a\theta)^n$, $n \in \mathbb{Z}$.

Using Lemma 1 and 2, it is sufficient to prove that ε^{-n} cannot be a binomial unit for $n \in \mathbb{N}$. It is easy to calculate an inverse unit ε^{-1} , $\varepsilon^{-1} = (1 + 12a^3) + a\theta + a^2\theta^2$.

Put $\varepsilon^{-n} = \alpha_n + \beta_n\theta + \gamma_n\theta^2$, where $\alpha_0 = 1$, $\beta_0 = 0$, $\gamma_0 = 0$, $\alpha_1 = 1 + 12a^3$, $\beta_1 = a$, $\gamma_1 = a^2$.

LEMMA 3.

$$(\alpha_n, \beta_n, \gamma_n) = (1, 0, 0)M^n(a), \quad M(a) = \begin{pmatrix} 1 + 12a^3 & 12a^2 & 12a \\ a & 1 & 0 \\ a^2 & a & 1 \end{pmatrix}.$$

Received by the editors October 11, 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 12A30.

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0002-9939/86 \$1.00 + \$.25 per page

PROOF. Calculating $\varepsilon^{-(n+1)}$, we have

$$\begin{aligned}\varepsilon^{-(n+1)} &= \varepsilon^{-1}\varepsilon^{-n} \\ &= \{(1 + 12a^3) + a\theta + a^2\theta^2\}(\alpha_n + \beta_n\theta + \gamma_n\theta^2) \\ &= (a^2\alpha_n + a\beta_n + \gamma_n)\theta^2 + (a\alpha_n + \beta_n)\theta + (1 + 12a^3)\alpha_n + 12a^2\beta_n + 12a\gamma_n.\end{aligned}$$

Comparing coefficients of θ , θ^2 and constant term, we obtain

$$\begin{aligned}\alpha_{n+1} &= (1 + 12a^3)\alpha_n + 12a^2\beta_n + 12a\gamma_n, \\ \beta_{n+1} &= a\alpha_n + \beta_n, \\ \gamma_{n+1} &= a^2\alpha_n + a\beta_n + \gamma_n,\end{aligned}$$

or in matrix expression, $(\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}) = (\alpha_n, \beta_n, \gamma_n)M(a)$. Therefore,

$$(\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}) = (1, 0, 0)M^{n+1}(a).$$

PROOF OF THE THEOREM. Generally we have to calculate $M^n(a)$ explicitly, using, for example, the Jordan canonical form of $M(a)$. However, in our case, it is enough to check that $\alpha_n, \beta_n, \gamma_n \neq 0$ for $n \geq 1$. All entries of the first column of $M^n(a)$ are greater than $1 + 12a^3$ for $n \geq 2$, so we obtain $\alpha_n, \beta_n, \gamma_n > 0$.

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