ON SQUARE ROOTS OF THE UNIFORM DISTRIBUTION ON COMPACT GROUPS

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ABSTRACT. Let G be a compact separable topological group. When does there exist a probability P such that $P * P = U$, where U is Haar measure and $P \neq U$? We show that such square roots exist if and only if G is not abelian, nor the product of the quaternions and a product of two element groups. In the course of proving this we classify compact groups with the property that every closed subgroup is normal.

1. Introduction. Let G be a compact separable topological group. When does there exist a probability P such that $P * P = U$ where U is Haar measure and $P \neq U$? Our main result is

THEOREM 1. *There is a probability P such that $P * P = U$ if and only if G is a nonabelian group which is not isomorphic to a product $\mathbf{H} \times E$ with $\mathbf{H}$ the eight element group of quaternions and E a product of two element groups.*

A proof of Theorem 1 appears in §2. The proof depends on the following result which is proved in §3.

THEOREM 2. *Let G be a compact, separable group with the property that every closed subgroup is normal. Then $G \simeq \mathbf{H} \times E \times O$ where $\mathbf{H}$ is the eight element group of quaternions, E is a product of two element groups, and O is a compact abelian group with Pontryagin dual a torsion group in which every element has odd order. The converse is also true.*

REMARK 1. Recall that a group is called Hamiltonian if every subgroup is normal. Dedekind and Baer characterized Hamiltonian groups as groups which can be represented as $\mathbf{H} \times E \times \tilde{O}$ with $\mathbf{H}$ and E as in Theorem 2, and $\tilde{O}$ a torsion group in which every element has odd order. Thus there is a 1-1 correspondence between Hamiltonian groups with E a countable product of two element groups and $\tilde{O}$ countable, and compact separable groups with every subgroup normal.

The countable torsion groups $\tilde{O}$ can be classified by using results in Kaplansky (1952). First, any torsion group is a direct sum of primary groups, and $\tilde{O}$ can have no 2-primary part. Then, Ulm's theorem gives a complete characterization of the other possible primary parts.

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REMARK 2. The problem studied here arose in a statistical context. One common method for generating uniform random variables on groups involves factoring the uniform distribution. Discussion and examples are in Chapter 4 of Diaconis (1982). Theorem 1 represents a first step in understanding such factorizations.

Theorem 1 is also related to problems of estimating the speed of convergence of random walks to Haar measure. Let G be a finite group of cardinality $|G|$. For P a probability on G , and U the uniform distribution, define the variation distance between P and U as

$$\|P - U\| = \sum |P(g) - U(g)|.$$

Aldous and Diaconis have shown that for most probabilities P (in the sense of the uniform distribution on the $|G|$ simplex) $\|P * P - U\| = o(1)$ as $|G|$ tends to infinity.

2. Proof of Theorem 1. We first introduce some notation and definitions. By a representation of a compact group G we mean a continuous homomorphism ρ of G into the group of invertible linear operators on a complex vector space V of dimension d_ρ . A representation ρ is irreducible if the only proper invariant subspace of V is $\{0\}$. Without loss of generality we assume throughout that all the irreducible representations are given by unitary matrices. For a representation ρ , its contragredient $\tilde{\rho}$ is defined by

$$\tilde{\rho}(g) = \rho(g^{-1})'$$

where $'$ denotes transpose. Then

$$\tilde{\rho}(g) = \overline{\rho(g)}.$$

The Fourier transform of a measure P on G is defined by

$$\rho(P) = \int_G \rho(g) P(dg).$$

Similarly, one defines the Fourier transform of a continuous function f on G . Then we have the Fourier inversion formula

$$f(g) = \sum_{\rho \in \hat{G}} d_\rho \operatorname{Tr}(\rho(g)^* \rho(f)).$$

Where $*$ denotes transpose of complex conjugate, $\hat{G}$ is the set of irreducible representations of G , and Haar measure on G is normalized so that G has total mass 1.

On a compact abelian group the factorization $U = P * P$ is impossible unless $P = U$. This follows because all irreducible representations are one dimensional and, for nontrivial ρ ,

$$0 = \rho(U) = \rho(P * P) = \rho(P)^2$$

implies $\rho(P) = 0$.

For nonabelian groups, the proof requires some preliminary lemmas.

LEMMA 1. *Let μ be a bounded measure on a compact group G . Then μ is real if and only if for every irreducible representation ρ of G , $\tilde{\rho}(\mu) = \overline{\rho(\mu)}$.*

PROOF. If μ is real, then

$$\tilde{\rho}_{ij}(\mu) = \int \bar{\rho}_{ij}(g) \mu(dg) = \overline{\rho_{ij}(\mu)}.$$

Conversely, suppose μ is a measure such that $\rho(\mu) = \overline{\rho(\mu)}$. This means

$$0 = \int \bar{\rho}_{ij}(g) \mu(dg) = - \int \bar{\rho}_{ij}(g) \bar{\mu}(dg)$$

or

$$0 = \int \rho_{ij}(g) \bar{\mu}(dg) - \int \rho_{ij}(g) \mu(dg).$$

Since this holds for every irreducible ρ , the Peter-Weyl theorem implies that the set function $\bar{\mu} - \mu$ is zero, so μ is real. $\square$

LEMMA 2. Let G be a compact noncommutative group. Then the following conditions are equivalent:

- (a) There is a probability measure $P \neq U$ such that $P * P = U$.
- (b) There is an irreducible (complex) representation ρ of G such that the algebra

$$R_\rho = \left\{ \sum_{g \in G} R_\rho(g) \right\}$$

contains nilpotent elements.

PROOF. If $U = P * P$ then $\rho(P)^2 = 0$ and $\rho(P) \neq 0$ for some ρ because $P \neq U$. It is easy to see that $\rho(P) \in R_\rho$ and so R_ρ contains nilpotent elements. Conversely, let $\gamma_1 \in R_\rho$ be nilpotent. If $\gamma_1^n = 0$ and n is smallest such power, then set $\gamma = \gamma_1^{n-1}$. This is nonzero and $\gamma^2 = 0$. Define a continuous f on G as follows: Set for every irreducible representation π of G

$$\begin{cases} \pi(f) = 0 & \text{if } \pi \neq \rho \text{ or } \bar{\rho}, \\ \rho(f) = \gamma, \\ \bar{\rho}(f) = \bar{\gamma} & \text{if } \bar{\rho} \text{ is not equivalent to } \rho. \end{cases}$$

This defines a nonzero continuous function by the Fourier inversion theorem. By Lemma 1, f is real. Notice that if ρ is equivalent to $\bar{\rho}$, say $\bar{\rho}(g) = \overline{\rho(g)} = T\rho(g)T^{-1}$ (T is unitary), then

$$\bar{\rho}(f) = T\rho(f)T^{-1} = \sum c_g \overline{\rho(g)} = \bar{\gamma}$$

and the hypothesis of Lemma 1 is satisfied. Clearly $\pi(f)^2 = 0$ for every irreducible representation π of G . It follows that for $\varepsilon > 0$ sufficiently small $P = (1 + \varepsilon f(g)) dg$ is a probability measure satisfying $P * P = U$. $\square$

REMARK. The relation between the existence of nilpotent elements and commutativity of the group has been investigated by M. Behncke (1971).

It was argued above that abelian groups do not admit a nontrivial square root of the uniform distribution. In light of Lemma 2, the nonabelian compact separable groups with the property that $R_\rho(G)$ has no nilpotents must be classified.

Let $M(G)$ denote the algebra, under convolution, of real measures on G . The following lemma has been abstracted from Sehgal (1975):

LEMMA 3. *If $M(G)$ has no nilpotent elements then every closed subgroup of G is normal.*

PROOF. Observe first that if R is any ring with unit and no nilpotents, then an idempotent $e = e^2$ in R commutes with every element $r \in R$. In fact, the equation $[er(1 - e)]^2 = 0$ implies $er(1 - e) = 0$, so $er = ere$. Similarly, $re = ere = er$. Now let $R = M(G)$, let H be a closed subgroup of G , and let e be Haar measure on H normalized so that $\text{vol}(H) = 1$. Then e is an idempotent in $M(G)$. For $g \in G$ let δ_g be a point mass at g . Then $\delta_g * e * \delta_{g^{-1}} = e$ which implies H is normal. $\square$

To complete the proof of Theorem 1, map $M(G)$ into $R_\rho(G)$ by $\mu \rightarrow \rho(\mu)$. From the Peter-Weyl theory, the map

$$M(G) \rightarrow \prod_{\rho} R_{\rho}(G), \quad \mu \rightarrow \prod_{\rho} (\rho(\mu))$$

is injective. Since $R_{\rho}(G)$ contains no nilpotent elements, neither does $M(G)$ and by Lemma 3, G is of the form given by Theorem 2. If O is not trivial, choose a character χ taking at least one nonreal value. Let ρ be the irreducible representation of H given by

$$i \rightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad j \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $\chi \otimes 1 \otimes \rho$ is an irreducible two-dimensional representation, and $R_{\chi \otimes 1 \otimes \rho}(G)$ is the full 2×2 complex matrix algebra, which contains nilpotent elements. This completes the proof of Theorem 1.

3. Proof of Theorem 2.

DEFINITION. A topological group is Hamiltonian if every closed subgroup is normal.

Recall that a finite group G is Hamiltonian if and only if it is of the form $G = \mathbf{H} \times F$ where F is a finite abelian group with no element of order 4 (see Hall (1959)).

LEMMA 4. *Closed subgroups and quotient groups of Hamiltonian groups are Hamiltonian.*

PROOF. Clear.

LEMMA 5. *A compact noncommutative Lie group of $\dim \geq 1$ is not Hamiltonian.*

PROOF. Let G^0 be a connected component of the identity in G . If G is Hamiltonian then so is G^0 . If G^0 is not abelian then it contains closed nonnormal subgroups, e.g. a maximal torus. So we may assume G^0 is a torus T , and G/T is finite. Hence we have the exact sequence

$$0 \rightarrow T \rightarrow G \xrightarrow{\eta} \mathbf{H} \times F \rightarrow (1)$$

where F is a finite abelian group with no element of order 4. Let $G' = \eta^{-1}(\mathbf{H})$. Then we have the exact sequence

$$(1) \quad 0 \rightarrow T \rightarrow G' \rightarrow \mathbf{H} \rightarrow \{1\}.$$

Let $c \in H^2(\mathbf{H}, T)$ be the cocycle defining the extension (1). Since $\mathbf{H}$ has order 8, $8c = 0$ in $H^2(\mathbf{H}, T)$ (see e.g. Mac Lane (1975)). This means there is $f: \mathbf{H} \rightarrow T$ such that $8\bar{c} - \delta f = 0$ where δ is the coboundary operator for nonhomogeneous cochains and $\bar{c}: \mathbf{H} \times \mathbf{H} \rightarrow T$ is a representative for the cocycle c . Clearly there is $\phi: \mathbf{H} \rightarrow T$ such that $8\phi = f$. Now the cochain $c' = \bar{c} - \delta\phi$ is also a representative for c and $8c' = 0$, i.e., c' takes values in the subgroup $\mathcal{E}$ of elements of orders dividing 8 in T . Therefore we have the commutative, row and column exact diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{E} & \rightarrow & K & \rightarrow & \mathbf{H} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} \\ 0 & \rightarrow & T & \rightarrow & G' & \rightarrow & \mathbf{H} \rightarrow 0 \end{array}$$

where K is defined by the cocycle c' . The subgroup K is finite, therefore a closed subgroup of G' . From the finite case, K and therefore G' and so G cannot be Hamiltonian. $\square$

For a separable compact group G , the Peter-Weyl theorem implies there is a sequence of finite dimensional representations ρ_n ($n \in \mathbf{N}$) such that

$$\bigcap_n \text{Ker } \rho_n = \{e\} \quad \text{and} \quad \text{Ker } \rho_n \supset \text{Ker } \rho_{n+1}.$$

LEMMA 6. *If G is a compact separable Hamiltonian group, then $\rho_n(G)$ is finite and $G = \varprojlim \rho_n(G)$ where the projective limit is taken relative to the system $\{\rho_n(G)\}$ with the obvious maps $\rho_{n+1}(G) \rightarrow \rho_n(G)$.*

PROOF. If G is Hamiltonian, then $\rho_n(G)$ is a Hamiltonian compact Lie group, and therefore finite. We have the inverse system of exact sequences:

$$\begin{array}{ccccccc} (1) & \rightarrow & K_n & \rightarrow & G & \rightarrow & \rho_n(G) \rightarrow (1) \\ & & \uparrow & & \uparrow \text{id} & & \uparrow \\ (1) & \rightarrow & K_m & \rightarrow & G & \rightarrow & \rho_m(G) \rightarrow (1) \end{array}$$

for $n \leq m$ where $K_m = \text{Ker } \rho_m \rightarrow K_n$ is the inclusion etc. Since in the category of compact groups $\varprojlim$ of inverse systems of exact sequences is exact (Eilenberg-Steenrod (1952, Chapter 8)), we have the exact sequence

$$(1) \rightarrow \bigcap_n K_n \rightarrow G \rightarrow \varprojlim \rho_n(G) \rightarrow (1).$$

The hypothesis on ρ_n implies $\bigcap K_n = \{e\}$. $\square$

It is no loss of generality to assume $\rho_1(G) = \mathbf{H}$. So we have the exact sequence

$$(2) \quad (1) \rightarrow K_1 \rightarrow G \xrightarrow{\rho_1} \mathbf{H} \rightarrow (1),$$

when we have identified $\rho_1(G)$ with $\mathbf{H}$.

LEMMA 7. Let G be a compact separable Hamiltonian group, and π_m ($m \geq n$) be the natural projection $\pi_m: \rho_m(G) \rightarrow \rho_n(G)$. Then we can choose a splitting $\rho_n(G) = \mathbf{H} \times F_n'' \times F_n'$ when F_n'' is a product of $\mathbf{Z}_2$'s and F_n' is an abelian group of odd order in such a way that $\pi_m|_{\mathbf{H}} = \text{id}$.

PROOF. We construct the splitting inductively. The case $n = 1$ being obvious, we assume the splitting has been constructed up to n . Consider the canonical homomorphism

$$\pi_{n+1, n}: \rho_{n+1}(G) \rightarrow \rho_n(G) = \mathbf{H} \times F_n'' \times F_n'$$

and any decomposition

$$\rho_{n+1}(G) \simeq \mathbf{H}' \times F_{n+1}'' \times F_{n+1}'$$

where $\mathbf{H}' \simeq \mathbf{H}$, F_{n+1}'' is a product of $\mathbf{Z}_2$'s and F_{n+1}' is a finite abelian group of odd order. Choose $(q_\alpha, \eta_\alpha, 0) \in \mathbf{H}' \times F_{n+1}'' \times F_{n+1}'$ ($\alpha = 1, 2$) such that

$$\pi_{n+1, n}(q_1, \eta_1, 0) = (i, 0, 0), \quad \pi_{n+1, n}(q_2, \eta_2, 0) = (j, 0, 0).$$

Set $q_3 = q_1 q_2$, $\eta_3 = \eta_1 \eta_2$, then $\pi_{n+1, n}(q_3, \eta_3, 0) = (k, 0, 0)$. Now define a homomorphism

$$\Phi_{n+1}: \mathbf{H} \rightarrow \mathbf{H}' \times F_{n+1}'' \times F_{n+1}'$$

by

$$\begin{aligned} \Phi_{n+1}(\pm 1) &= (\pm e, 0, 0), & \Phi_{n+1}(\pm i) &= (\pm q_1, \eta_1, 0), \\ \Phi_{n+1}(\pm j) &= (\pm q_2, \eta_2, 0), & \Phi_{n+1}(\pm k) &= (\pm q_3, \eta_3, 0). \end{aligned}$$

The fact that Φ_{n+1} is a homomorphism can be checked by straightforward verification, e.g., let us show

$$(3) \quad \Phi_{n+1}(j) = \Phi_{n+1}(ki).$$

By construction $\Phi_{n+1}(j) = (q_2, \eta_2, 0)$,

$$\Phi_{n+1}(k)\Phi_{n+1}(i) = (q_3 q_1, \eta_3 \eta_1, 0)$$

and $\eta_3 \eta_1 = \eta_2$. Also

$$\pi_{n+1, n}(q_2, \eta_2, 0) = (j, 0, 0) = \pi_{n+1, n}(q_3 q_1, \eta_3 \eta_1, 0).$$

Hence $(q_3 q_1 q_2^{-1}, 0, 0) \in \text{Ker } \pi_{n+1, n}$. If $q_3 q_1 q_2^{-1} \neq e$ then $\text{Ker } \pi_{n+1, n}|_{\mathbf{H}'} \neq \{e\}$ and then $\text{im } \pi_{n+1, n}$ would be abelian. This proves (3). Let $\mathbf{H}'' = \text{im } \Phi_{n+1}$. We have the decomposition

$$\rho_{n+1}(G) = \mathbf{H}'' \times F_{n+1}'' \times F_{n+1}'.$$

Now notice that the projection $\pi_{n+1, n}|_{\mathbf{H}''}$ is simply the identity map after possibly relabelling. $\square$

LEMMA 8. Let G be a compact separable Hamiltonian group. Then the exact sequence (2) splits and furthermore $G \simeq \mathbf{H} \times K_1$ as a direct product.

PROOF. It suffices to prove the first assertion since if the sequence (2) splits and G is a semidirect product of K_1 and $\mathbf{H}$ which is not a direct product, then $\mathbf{H}$ would be a closed subgroup which is not normal. To prove that (2) splits, we have to construct

a homomorphism

$$\beta: \mathbf{H} \rightarrow G = \varprojlim \rho_n(G)$$

such that $\rho_1 \circ \beta = \text{id}_{\mathbf{H}}$. To do this it suffices to construct $\beta_n: \mathbf{H} \rightarrow \rho_n(G)$ such that

$$(4) \quad \begin{array}{ccc} & \rho_n(G) & \\ \beta_n \nearrow & & \\ \mathbf{H} & \uparrow \pi_{mn} & m \geq n \\ \beta_m \searrow & & \\ & \rho_m(G) & \end{array}$$

commutes and $\beta_1 = \text{id}$. We define $\beta_1 = \text{id}$. Consider the decomposition $\rho_n(G) = \mathbf{H} \times F_n'' \times F_n'$ provided by Lemma 7. Define

$$\beta_n(i) = (i, 0, 0), \quad \beta_n(j) = (j, 0, 0), \quad \text{etc.}$$

By Lemma 7, the commutativity condition (4) is satisfied. $\square$

We now complete the proof of Theorem 2. We necessarily have $\pi_{n+1,n}(F_{n+1}') \subset F_n'$ and $\pi_{n+1,n}(\mathbf{H} \times F_{n+1}'') \subset \mathbf{H} \times F_n''$. Hence

$$G = \varprojlim (\mathbf{H} \times F_n'') \times \varprojlim (F_n').$$

It remains to show

$$(5) \quad \varprojlim (\mathbf{H} \times F_n'') = \mathbf{H} \times \varprojlim (F_n'')$$

where limits are taken with respect to the obvious maps. By definition

$$\varprojlim (\mathbf{H} \times F_n'') = \{((q, f_1), (q, f_2), \dots) \mid \pi_{mn}((q, f_m)) = (q, f_n)\}.$$

Now $\pi_{mn}(q, 0) = (q, 0)$, hence if $((q, f_1), (q, f_2), \dots) \in \varprojlim (\mathbf{H} \times F_n'')$ we have

$$(6) \quad \pi_{mn}(e, f_m) = (e, f_n).$$

Conversely, if (6) holds then $((q, f_1), (q, f_2), \dots) \in \varprojlim (\mathbf{H} \times F_n'')$. This proves (5) and Theorem 2 with O presented as an abelian profinite group. Shatz (1972, p. 10) shows that an abelian group is profinite if and only if its dual is a torsion group. $\square$

In conclusion we note that a compact Hamiltonian group does not necessarily have the property that every subgroup is normal. In fact, $\mathbf{H} \times \prod \mathbf{Z}_p$ ($\mathbf{Z}_p$ = integers mod prime p) is Hamiltonian in our sense, however, the cyclic subgroup generated by $(i, 1, 1, \dots)$ is not normal.

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