A PROPERTY OF IDEALS IN POLYNOMIAL RINGS

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ABSTRACT. Every ideal in the polynomial ring in n variables over an infinite field has a reduction generated by n elements.

Eisenbud and Evans [2] proved that every ideal in $k[X_1, \dots, X_n]$ can be generated up to radical by n elements (where k is a field). Avinash Sathaye [7] and Mohan Kumar [5] proved a locally complete intersection in $k[X_1, \dots, X_n]$ can be generated by n elements.

In this short note we show that every ideal in $k[X_1, \dots, X_n]$ has a nice approximation generated by n elements. More precisely, we prove the following.

THEOREM. *Let k be an infinite field. Then every ideal I in $k[X_1, \dots, X_n]$ has a reduction J generated by n elements.*

By [6], J is a reduction of I if there exists an integer r such that $JI^r = I^{r+1}$. Northcott and Rees [6] point out that J can be regarded as a simplified version of I preserving many properties of I , in particular the multiplicities at minimal prime over-ideals. Moreover, J has the same radical as I and if I is locally a complete intersection, then it is the only reduction of itself, hence a connection between our result and those of Eisenbud and Evans and Sathaye and Kumar.

PROOF OF THE THEOREM. Since $k[X_1, \dots, X_n]$ is a UFD, we can assume that $n \geq \dim(A/I) + 2$. Let $g_1, \dots, g_r$ be a system of generators of I . Set $A = k[X_1, \dots, X_n]$ and $B = k[g_1, \dots, g_r] \subset A$. The dimension of B is at most n , since its quotient field is a subfield of $k(X_1, \dots, X_n)$ and therefore has transcendence degree $\leq n$. Denote by P the ideal of B generated by $g_1, \dots, g_r$. Let $\bar{P}$ be the image of P in B_P , the localization of B at P (i.e. $\bar{P}$ is the maximal ideal of B_P). Since B_P is local of dimension $\leq n$, by Burch [1] $\bar{P}$ has a reduction $\bar{Q}$ generated by n elements $h_1, \dots, h_n$. Let r be an integer such that $\bar{Q}\bar{P}^r = \bar{P}^{r+1}$. Since $\bar{Q}$ is $\bar{P}$ -primary, there exists a unique P -primary ideal $Q \subset B$, the image of which in B_P coincides with $\bar{Q}$. Since P is a maximal ideal of B , we see that the ideals QP^r and P^{r+1} are P -primary and their images in B_P coincide. Therefore, $QP^r = P^{r+1}$ in B , since there is a one-to-one correspondence between the $\bar{P}$ -primary ideals of B_P and the P -primary ideals of B .

Let J be the extension of Q in A . Since the extension of P is I and $QP^r = P^{r+1}$, we see that $JI^r = I^{r+1}$, i.e. J is a reduction of I .

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Let $s \in B \setminus P$ be an element such that $Q_s = (h_1, \dots, h_n)$. Thus $J_s = (h_1, \dots, h_n)$. Since Q is P -primary and P is a maximal ideal of B , we see that $(s) + Q = (1)$, i.e., regarding s as an element of A , that $(s) + J = (1)$. The conditions $(s) + J = (1)$ and $J_s = (h_1, \dots, h_n)$ imply that J/J^2 is generated by n elements $h_1, \dots, h_n$. Now Theorem 5 of Mohan Kumar [4] tells us that J is generated by n elements, since $n \geq \dim(A/J) + 2$. Q.E.D.

REMARKS. 1. In a similar way one can prove that every ideal in a finitely generated n -dimensional algebra over an infinite field has a reduction generated by $n + 1$ elements.

2. For every prime ideal $P \subset B$ containing I we have the dimension of

$$(J + P \cdot I)/P \cdot I$$

as a vector space over the quotient field of A/P is $\geq$ height I . This improves Eisenbud and Evans [2, 3] who proved only that $J \not\subset P \cdot I$.

3. Some generalization of our theorem is possible to the case $I \subset A[X]$, where A is an $(n - 1)$ -dimensional finitely generated algebra over an infinite field. For this one has to use S. Mandall's extension of Mohan Kumar's theorem to ideals in $A[X]$ (cf. [5]).

CONJECTURE. Let A be a commutative Noetherian ring of dimension $n - 1$ such that the residue field of every maximal ideal of A is infinite. Let I be an ideal of A or $A[X]$. Then I has a reduction generated by n elements.

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