ON NILPOTENCY IN THE ADO-HARISH-CHANDRA THEOREM ON LIE ALGEBRA REPRESENTATIONS

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ABSTRACT. Let L be a finite-dimensional Lie algebra, over an arbitrary field, and regard L as embedded in its enveloping algebra UL .

THEOREM. *If K is an ideal of L and K is nilpotent of class q , then for any r there exists a finite-dimensional representation ρ of L which vanishes on all products (in UL) of $\geq qr + 1$ elements of K and is faithful on the subspace of UL spanned by all products of $\leq r$ elements of L .*

This result sharpens (with respect to nilpotency) the Ado-Iwasawa theorem on the existence of faithful representations and the Harish-Chandra theorem on the existence of representations separating points of UL .

This paper gives a sharpened (with respect to nilpotency) form of two classical theorems: Ado's theorem on the existence of a faithful representation of a Lie algebra, and the related theorem of Harish-Chandra on the existence of a representation not annihilating a given element of the universal enveloping algebra. Our results are valid both at characteristic 0 and at prime characteristic, although our proofs are different for the two cases.

Throughout the paper L denotes a finite-dimensional Lie algebra, regarded as embedded in its enveloping algebra UL , and N denotes the largest nilpotent ideal of L . Representation of L will always mean finite-dimensional representation and, when convenient, the same notation will be used for a representation of L and for the corresponding extension to a representation of UL .

We recall that Ado's theorem has a shorter version and a more precise version which are, respectively, as follows.¹

THEOREM A. *L has a faithful representation.*

THEOREM B. *L has a faithful representation ρ which is nilpotent on N (that is, $\rho(x)$ is nilpotent for all $x \in N$).*

By Engel's theorem, the nilpotency of ρ on N is equivalent to the existence of an $m > 0$ such that $\rho(x_1 \cdots x_m) = 0$ for all $x_1, \dots, x_m \in N$. In this paper we shall show how small a value of m can be attained. In fact we shall prove the following

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¹At characteristic 0, these results are known as Ado's theorem; E. Cartan, Hochschild, Iwasawa, Harish-Chandra, Zassenhaus, and Hochschild and Mostow also contributed proofs of Theorem A or B in this case. At prime characteristic, Theorem A is a result of Iwasawa, with a short proof having been given by Jacobson (see Chapter 6, "The theorem of Ado-Iwasawa," of [Ja2].) The prime characteristic version of Theorem B is covered by Hochschild [Ho2].

result (recall that the nilpotency class of a nilpotent Lie algebra K is the smallest q such that all bracket products of $q + 1$ elements of K are 0, or equivalently, such that $(\text{ad}_K K)^q = 0$).

THEOREM 1. *Suppose K is a nilpotent ideal (of L) of nilpotency class q . Then there exists a faithful representation ρ of L which satisfies $\rho(x_1 \cdots x_{q+1}) = 0$ for all $x_1, \dots, x_{q+1} \in K$.*

Here the number $q + 1$ of factors is the best possible, since if all products of q elements of $\rho(K)$ are zero, then also all bracket products of q elements of $\rho(K)$ are zero, and so by faithfulness K has class $< q$, a contradiction. The result of Theorem 1 appears to be new even for $K = N$.

We next recall a result of Harish-Chandra, which is the main theorem of [Ha].²

THEOREM C. *If $0 \neq u \in UL$ then there exists a representation ρ of L such that $\rho(u) \neq 0$.*

For $i \geq 0$ let $(UL)_i$ denote the subspace of UL spanned by all products of $\leq i$ elements of UL ; thus $\{(UL)_i\}_{i \geq 0}$ is the canonical ascending filtration of UL . It is pointed out in [Ha] that the statement of Theorem C (whose proof uses Theorem A) in fact implies Theorem A. We note that the same argument implies the following formal strengthening of Theorem C.

THEOREM D. *For any $r \geq 0$ there exists a representation of L which is faithful on $(UL)_r$.*

As our main result, we shall prove the following theorem, which extends all of the above and is valid at arbitrary characteristic.

THEOREM 2. *Suppose K is a nilpotent ideal (of L) of nilpotency class q . For any $r \geq 0$ there exists a representation ρ of L which is faithful on $(UL)_r$ and satisfies $\rho(x_1 \cdots x_{qr+1}) = 0$ for all $x_1, \dots, x_{qr+1} \in K$.*

Again the number $qr + 1$ of factors is the best possible, since if x is a nonzero bracket product of q elements of K , then $0 \neq \rho(x^r) \in \rho(K)^{qr}$.

We shall proceed to the proof, using the following notation: K denotes a nilpotent ideal (of L) of nilpotency class q ; we may assume that $K \neq 0$. The terms of the lower central series of K will be denoted by K_i , where $K_1 = K$ and $K_{i+1} = [K, K_i]$; thus $K_{q+1} = 0$ and $K_q \neq 0$. Also, for each i , K^i will denote the span in UL of $KK \cdots K$ (i times). Note that $K_i \subseteq K^i$. We define an integer-valued function w on $K - \{0\}$ by $w(x) = n$ if $x \in K_n - K_{n+1}$. We take a basis $\{x_1, \dots, x_s\}$ of K which is compatible with the lower central series of K , that is, for each $i = 1, \dots, q$ there is an $l_i \in \{1, \dots, s\}$ such that $\{x_k | l_i \leq k \leq s\}$ is a basis of K_i . Let S_k denote the span of

$$\left\{ x_{j_1} \cdots x_{j_n} \mid n \geq 1, 1 \leq j_1 \leq \cdots \leq j_n \leq s, \sum_{i=1}^n w(x_{j_i}) = k \right\}.$$

²Expositions of the theorem are also given in [D1] and in exercises in [Bo, Ja2]. Again, this theorem was only given at characteristic 0, but its validity at prime characteristic follows from a modification of the proof (or extension) of the Ado-Iwasawa theorem by Jacobson [Ja, Ja2] or Hochschild [Ho2]. W. Michaelis has announced a coalgebra proof of the Harish-Chandra theorem at arbitrary characteristic.

LEMMA 1. For any $k \geq 1$,

$$\sum_{j \geq k} K^j = \sum_{j \geq k} S_j.$$

PROOF. Since $K_i \subseteq K^i$, $S_k \subseteq K^k$. In the process of straightening a monomial in $x_1, \dots, x_s$ to express it as a linear combination of standard monomials, each straightening gives a linear combination of terms $x_{m_1} \cdots x_{m_t}$ for which $\sum_{i=1}^t w(x_{m_i})$ has at least the previous value, since $[K_i, K_j] \subseteq K_{i+j}$. Hence $K^j \subseteq \sum_{j \leq i} S_i$.

We now prove Theorem 1 under the assumption that the characteristic is 0. Let zL denote the center of L . Then $K + zL$ is also nilpotent of class q , and so without loss of generality we may assume that $zL \subseteq K$. Let I denote the ideal of UK generated by K^{q+1} . Then $(UK)/I$ is finite dimensional and so has a faithful (finite-dimensional) representation τ , say in V . It follows from Lemma 1 that $I \cap K = 0$. Hence the composite mapping

$$K \rightarrow UK \rightarrow (UK)/I \xrightarrow{\tau} \text{End } V$$

gives a faithful representation σ of K in V , and $\sigma(K)^{q+1} = 0$ since $K^{q+1} \subseteq I$.

We now use Proposition 7.3.1 of [Bo] to obtain an aggrandizement ρ of σ to L (that is, a representation ρ of L , say in V' , for which there exists an injective K -module map of V into V') which is nilpotent on N . The proof in [Bo] obtains ρ by a chain of aggrandizements, where at each step the result follows from the aggrandizement theorem of Zassenhaus (Theorem 7.2.1 of [Bo]). We shall now observe that in each of these applications of Zassenhaus' theorem, the aggrandizement annihilates K^{q+1} . In each case we are dealing with a subalgebra $H = M + H'$ of L which is a vector space direct sum of a subalgebra M and an ideal H' of H with $K \subseteq H'$, and a representation ρ' of H' (acting say in W) having the desired nilpotency properties. To obtain an aggrandizement of ρ' to H , one starts with the sub- H -module S of $(UH')^*$ generated by the space

$$C(\rho') = \{\lambda \circ \rho' \mid \lambda \in (\text{End } W)^*\},$$

of representative functions of UH' corresponding to the representation ρ' , where if $y \in M$, then y acts on $(UH')^*$ via $yf(u) = f(uy - yu)$ and if $y \in H'$ then y acts via $yf(u) = f(uy)$. In [Bo] it is shown that for some n , $S \oplus \cdots \oplus S$ (n times) furnishes the desired aggrandizement. In our case it thus suffices to show that $K^{q+1}S = 0$. Since $\rho'(K)^{q+1} = 0$, we have $K^{q+1}C(\rho') = 0$. Also $K^{q+1}(UM) \subseteq (UM)K^{q+1}$. Therefore

$$\begin{aligned} K^{q+1}S &= K^{q+1}(UM)(UH')C(\rho') \\ &= K^{q+1}(UM)C(\rho') \subseteq (UM)K^{q+1}C(\rho') = 0. \end{aligned}$$

Thus the aggrandizement of σ to L is a representation which is nilpotent on N , annihilates K^{q+1} , and is faithful on K (since σ is faithful). Adding to this representation the adjoint representation on L (which also annihilates K^{q+1}), we get the desired faithful (since $zL \subseteq K$) representation of L . This completes the proof of Theorem 1 at characteristic 0.

We now prove Theorem 2 at characteristic 0. Let $y_1, \dots, y_n$ be a basis of L and $b_1, \dots, b_n$ the dual basis of L^* . Let ρ (acting in V) be a representation of

L satisfying the conclusion of Theorem 1. If we regard L as embedded (by ρ) in $\text{End } V$, then any linear functional on L can be extended to a linear functional on $\text{End } V$. Hence for $i = 1, \dots, n$, there exists $\lambda_i \in (\text{End } V)^*$ such that $\lambda_i \circ \rho = b_i$ (on L). We write $\lambda_i \circ \rho = f_i$, regarded as a linear functional on UL . By taking (if necessary) a direct sum of ρ with a trivial representation, we may assume that $f_i(1) = 0$, $i = 1, \dots, n$.

The f_i are elements of $R(L)$, the space of representative functions of UL ($R(L)$ is the sum, over all representations π , of the corresponding space $C(\pi)$ of functions for the representation π ; $R(L)$ is also denoted $(UL)^0$ —see [Sw]). We recall [Ho] that $R(L)$ is a commutative algebra (in fact a Hopf algebra) with multiplication given by

$$(\lambda_1 \circ \pi_1)(\lambda_2 \circ \pi_2) = (\lambda_1 \otimes \lambda_2) \circ (\pi_1 \otimes \pi_2),$$

and unit element ε given by $\varepsilon(1) = 1$, $\varepsilon(L \cdots L) = 0$ (i factors of L) for $i \geq 1$. For $f \in R(L)$, we write $f_j f$ for the divided power $f^j/j!$.

We use the basis of UL consisting of the standard monomials

$$\{y_1^{t_1} \cdots y_n^{t_n} | t_1, \dots, t_n \geq 0\}.$$

Since $f_i(y_i) = \delta_{ij}$ and $f_i(1) = 0$, it follows from the formula for the tensor product of representations of UL that

$$((\gamma_{s_1} f_1) \cdots (\gamma_{s_n} f_n))(y_1^{s_1} \cdots y_n^{s_n}) = 1$$

and that $(\gamma_{s_1} f_1) \cdots (\gamma_{s_n} f_n)$ annihilates all other standard monomials of degree $\leq s_1 + \cdots + s_n$. Suppose $r > 0$ and $x \in (UL)_r - (UL)_{r-1}$. Then

$$(\gamma_{s_1} f_1) \cdots (\gamma_{s_n} f_n)x \neq 0$$

for some $s_1, \dots, s_n$ with $s_1 + \cdots + s_n = r$. We write $\pi = \bigotimes^r \rho$ ($= \rho \otimes \cdots \otimes \rho$ (r times))). Then

$$(\gamma_{s_1} f_1) \cdots (\gamma_{s_n} f_n)x = \left((1/s_1! \cdots s_n!) \left(\left(\bigotimes^{s_1} \lambda_1 \right) \otimes \cdots \otimes \left(\bigotimes^{s_n} \lambda_n \right) \right) \circ \pi \right) x.$$

Therefore $\pi x \neq 0$. Hence $(\ker \pi) \cap (UL)_r \subseteq (UL)_{r-1}$. Since $\rho(K^{q+1}) = 0$, we have $\pi(K^{qr+1}) = 0$. Since ρ is nilpotent on N so is π . By induction on r (for $r = 0$ one can use a trivial representation), there is a representation π' of L faithful on $(UL)_{r-1}$, with $\pi'(N)$ nilpotent and $\pi'(K^{q(r-1)+1}) = 0$. Hence $\pi'(K^{qr+1}) = 0$, and $\pi \oplus \pi'$ has the required properties. This completes the proof of Theorem 2 at characteristic 0.

REMARK. The above proof actually shows the following. Let ρ be a faithful representation of L , and define ρ' to be either ρ or the sum of ρ and the trivial representation accordingly as $\rho(L)$ does not or does contain the identity. Then

$$\rho' \oplus (\rho' \otimes \rho') \oplus \cdots \oplus \left(\bigotimes^r \rho' \right)$$

is faithful³ on $(UL)_r$ (and annihilates K^{qr+1} if ρ annihilates K^{q+1}).

Our representation has been chosen so that it is also nilpotent on N . By a result of Hochschild [Ho2], this has the following consequence.

³The proof of this can be extended to prime characteristic by using the divided powers in $R(L)$ in that case. For the existence of these divided powers, see [Bl].

COROLLARY 1. *At characteristic 0 there exists a representation as in Theorem 2 with the additional property that it is nilpotent on each $x \in L$ for which $\text{ad } x$ is nilpotent.*

We now prove Theorem 2 in the characteristic p case. We extend the basis $\{x_1, \dots, x_s\}$ of K to a basis $\{x_1, \dots, x_n\}$ of L . By Lemma 6.5 of [Ja2], for each i there exists a p -polynomial $m_i(\lambda)$, of degree d_i , such that $z_i = m_i(x_i)$ is in the center of UL . We may assume that $d_i > r$. We write

$$B = \{z_{s+1}^{h_{s+1}} \cdots z_n^{h_n} x_1^{j_1} \cdots x_n^{j_n} | h_i, j_i \geq 0, j_i < d_i \ (l > s)\}.$$

Then B is a basis of UL (cf. Lemma 5.4 of [Ja2]). For $b \in B$ we denote the corresponding h_i (with $i > s$) by $h_i(b)$ and $\sum_{i=1}^s j_i w(x_i)$ by $w(b)$. We set

$$B' = \{b \in B | w(b) > qr \text{ for some } h_i(b) > 0\},$$

and let J be the ideal of UL generated by $z_{s+1}, \dots, z_n$ and K^{qr+1} . Then

$$J = (UL)K^{qr+1}(UL) + \sum_{i>s} z_i(UL) = K^{qr+1}(UL) + \sum_{i>s} z_i(UL).$$

Also $B' \subseteq J$, and K^{qr+1} is in the span of B' , by Lemma 1. If $b \in B$ then $z_i b \in B'$ for $i > s$, and $K^{qr+1}b$ is in the span of the elements y of B' with $w(y) > qr$, by Lemma 1. It follows that $z_i(UL)$ and $K^{qr+1}(UL)$ are contained in the span of B' . Therefore B' is a basis of J , the cosets of the elements of $B - B'$ form a basis of $(UL)/J$, and $(UL)/J$ is finite dimensional. The standard monomials (in $x_1, \dots, x_n$) of degree $\leq r$ form a basis of $(UL)_r$. Every such monomial is in $B - B'$ since each $d_i > r$ ($i > s$) and since $w(x_i) \leq q$ ($i \leq s$). Hence the map $UL \rightarrow (UL)/J$ is faithful on $(UL)_r$. Composing this with a faithful representation of $(UL)/J$, we get a representation ρ of UL which is faithful on $(UL)_r$. Moreover $\rho(K)^{qr+1} = 0$. $\square$

The following is a sharpened version, for algebras containing a nonzero nilpotent ideal, that is, for nonsemisimple algebras, of the existence of indecomposable representations of arbitrarily high degree.

COROLLARY 2. *Suppose K is an ideal of L , K is nilpotent of class q , $0 \neq x \in K$, and $r > 0$. Then there exists an indecomposable representation σ of L such that $\sigma(x)^r \neq 0$ but $\sigma(x)^{qr+1} = 0$.*

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