HARNACK'S INEQUALITY FOR SCHRÖDINGER OPERATORS AND THE CONTINUITY OF SOLUTIONS

F. CHIARENZA, E. FABES AND N. GAROFALO

ABSTRACT. We prove a uniform Harnack inequality for nonnegative solutions of $Au = Vu$, where A is a second order elliptic operator in divergence form, and V belongs to the Stummel class of potentials. As a consequence we obtain the continuity of a general weak solution. These results extend the previous work of Aizenman and Simon for $\Delta u = Vu$.

Introduction. In this paper we consider Schrödinger operators of the form

$$Au - Vu \equiv \sum_{i,j=1}^n D_{x_i} (a_{ij}(x) D_{x_j} u(x)) - V(x)u(x),$$

where the matrix $a(x) = (a_{ij}(x))$ is symmetric, bounded, measurable, and positive definite uniformly in x . This means, in particular, that there is a number $\lambda > 0$ such that, for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$,

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \lambda^{-1} |\xi|^2.$$

The function V is assumed to belong to the Stummel class of potentials, i.e., for each bounded set $\Omega \subset \mathbb{R}^n$,

$$(*) \quad \lim_{r \downarrow 0} \sup_{x \in \Omega} \int_{|x-y| < r} \frac{|V(y)|}{|x-y|^{n-2}} dy = 0 \quad (n \geq 3).$$

The main results establish a local (uniform) Harnack inequality and a local modulus of continuity for weak solutions of $Au - Vu = 0$ in Ω . A weak solution of $Au - Vu = 0$ in Ω is a function u in $H_{loc}^1(\Omega) = \{u \in L_{loc}^2(\Omega): \nabla u \in L_{loc}^2(\Omega)\}$ satisfying

$$-\int a(x)(\nabla u(x)) \cdot \nabla \varphi(x) dx = \int V(x)u(x)\varphi(x) dx$$

for any $\varphi \in C_0^\infty(\Omega)$.

The dependence of the main results on V is determined by the rate of convergence to zero of the

$$\sup_{x \in \Omega} \int_{|x-y| < r} \frac{|V(y)|}{|x-y|^{n-2}} dy.$$

Received by the editors September 19, 1985.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 35B45, 35B65, 35J10.

Key words and phrases. Harnack's inequality, Schrödinger equation.

©1986 American Mathematical Society
0002-9939/86 \$1.00 + \$.25 per page

We make this precise by fixing a nondecreasing function $\eta(r)$, $r > 0$, such that $\lim_{r \downarrow 0} \eta(r) = 0$. We set

$$K_\eta = \left\{ V: \sup_{x \in \Omega} \int_{|x-y| \leq r} \frac{|V(y)|}{|x-y|^{n-2}} dy \leq \eta(r), \text{ for } r > 0 \right\}.$$

The primary theorems can be stated as follows:

HARNACK'S THEOREM. *There exist positive constants r_0 and C , depending only on λ , n , and η such that for any nonnegative solution u of $Au - Vu = 0$ in Ω and, for any ball B_r with $0 < r \leq r_0$ and $B_{2r} \subset \Omega$, we have $\sup_{B_{r/2}} u \leq C \inf_{B_{r/2}} u$.*

CONTINUITY THEOREM. *There exists a nondecreasing function $\omega(s)$ depending only on λ and η such that $\omega(s) \rightarrow 0$ as $s \rightarrow 0$ and, for any solution u of $Au - Vu = 0$ in Ω , and for any ball $B_r(x_0)$ with $B_{3r}(x_0) \subset \Omega$, we have*

$$|u(x) - u(x_0)| \leq \omega\left(\frac{|x - x_0|}{r}\right) \sup_{B_{3r}(x_0)} |u|, \quad x \in B_r(x_0).$$

The above results for the case $A = \Delta$ were recently proved by Aizenman and Simon in their celebrated work [1] (see also [14]). Our technique seems quite different. It is nonprobabilistic, and for the Harnack inequality is based on a real variable approach found in [5]. It is, in fact, a combination of ideas originally due to Krylov and Safanov [11] and Trudinger [13].

Before proceeding to the main body of the paper, we want to remark, without proof, that we may assume the matrix above and the potential $V(x)$ are smooth. Under our hypotheses, regularizations of these functions do not change the structural elements of our set-up, namely λ and η . Weak solutions of $(A - V)u = 0$ in Ω are locally limits in L^2 of solutions of equations with such regularized $a(x)$ and $V(x)$. Hence we may, and henceforth do, assume these functions are smooth.

Finally, the notation we will use is standard and should be clear in context. Dependence of constants on λ and/or η will be so indicated, while the dependence on dimension, though pervasive, will not be indicated.

All our results are stated and proved for the case when the dimension $n \geq 3$. It is not difficult to see that solutions of $Au = Vu$ in $\Omega \subset \mathbb{R}^2$ with V satisfying

$$\sup_{x \in \Omega} \int_{|x-y| < r} \log|x-y| |V(y)| dy \leq \eta(r)$$

are also solutions of a three-dimensional problem. Hence our results remain valid for $n = 2$.

1. The L^∞ -bounds. We begin with a "Caccioppoli-type" estimate.

LEMMA 1.1. *Suppose u satisfies $Au = Vu$ in Ω . If $0 < s < t$ and $B_t \subset \Omega$ then*

$$\int_{B_s} |\nabla u|^2 \leq c(\lambda, \eta) \frac{1}{(t-s)^2} \int_{B_t} u^2.$$

PROOF. Let $\phi \in C_0^\infty(B_t)$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ on B_s , and $|\nabla \phi|^2 \leq c/(t-s)$. Then

$$\begin{aligned} \int |\nabla u|^2 \phi^2 &\leq c \int a(\nabla u) \cdot \nabla u \phi^2 \leq c \int a(\nabla u) \cdot \nabla (u \phi^2) - 2c \int a(\nabla u) \cdot \nabla \phi \phi u \\ &\leq -c \int V u^2 \phi^2 - 2c \int a(\nabla u) \cdot \nabla \phi \phi u. \end{aligned}$$

From the imbedding properties of $|V|$ [12, p. 138], for any $\varepsilon > 0$,

$$\int |V| u^2 \phi^2 \leq \varepsilon \int |\nabla u|^2 \phi^2 + \frac{C_\varepsilon}{(t-s)^2} \int_{B_t} u^2.$$

Hence

$$\int_{B_s} |\nabla u|^2 \phi^2 \leq \frac{C}{(t-s)^2} \int_{B_t} u^2.$$

The “Caccioppoli-type” inequality allows us to compare mean values of powers of solutions. We will use the standard notation $f_B f$ for the mean value $(1/|B|) \int_B f$ of f over B .

LEMMA 1.2. *There exists a constant $C(\lambda, \eta)$ such that if u satisfies $Au = Vu$ in Ω and B_{2r} is a ball $\subset \Omega$, then*

$$\left(\int_{B_{r/2}} u^2 \right)^{1/2} \leq C \int_{B_r} |u|.$$

PROOF. By translation we may assume the center of B_r is the origin. Set $u_r(x) = u(rx)$. u_r is a solution in $B_2 \equiv B_2(0)$ of $A_r u_r = V_r u_r$ where $A_r = \sum_{i,j=1}^n D_{x_i}(a_{ij}(rx)D_{x_j})$ and $V_r(x) = r^2 V(rx)$. We observe that

$$\sup_{x \in B_2} r^2 \int_{|y| < 2} \frac{|V(ry)|}{|x-y|^{n-2}} dy \leq \sup_{0 < \alpha < 4r} \sup_{w \in \Omega} \int_{B_\alpha(w)} \frac{|V(z)|}{|w-z|^{n-2}} dz.$$

Since r is a bounded quantity this last expression is bounded by $c(\eta)$. Hence to establish Lemma 1.2 we may take $r = 1$, $\Omega = B_2 \equiv B_2(0)$ and the center of B_1 and $B_{1/2}$ to be the origin. The succeeding argument is taken from [5, pp. 1004–1005].

We may assume $\int_{B_1} |u| = 1$. For $1/2 \leq s \leq 1$ set $I(s) = (\int_{B_s} u^2)^{1/2}$. We will show $I(1/2) \leq c$. If $I(1/2) \leq 1$, we are done, so let us assume $I(1/2) > 1$.

$$I(s) = \left(\int_{B_s} |u|^{2-\theta} |u|^\theta \right)^{1/2} \leq \left(\int_{B_s} |u|^{(2-\theta)/(1-\theta)} \right)^{(1-\theta)/2}.$$

Now choose θ so that $(2-\theta)/(1-\theta) = 2\tau$, where $\tau = n/(n-2)$. Then $\theta = (2\tau-1)/(2(\tau-1))$ and we have $I(s) \leq (\int_{B_s} |u|^{2\tau})^{(1-\theta)/2}$.

A combination of Lemma 1.2 and Sobolev's inequality imply, for $1/2 \leq s < t \leq 1$, $I(s) \leq c[(t-s)^{-1}I(t)]^{\tau(1-\theta)}$ and so

$$\log I(s) \leq \log c + \tau(1-\theta) \log(t-s)^{-1} + (\tau(1-\theta)) \log I(t).$$

Set $s = t^b$ with $b > 1$ to be chosen. Then

$$\int_{(1/2)^{1/b}}^1 \frac{\log I(t^b)}{t} dt \leq c + \tau(1 - \theta) \int_{(1/2)}^1 \frac{\log I(t)}{t} dt.$$

(Remember $I(1/2)x \geq 1$.) Making the change of variable $u = t^b$ we obtain

$$\left(\frac{1}{b} - \left(\frac{1 - \theta}{2} \right) \right) \int_{1/2}^1 \frac{\log I(t)}{t} dt \leq C.$$

We now choose $b > 1$ so that $1/b - \tau(1 - \theta) > 0$, and since $I(s)$ is increasing, $\log I(1/2) \leq C$.

The next lemma establishes the existence and integrability properties of the Green's function for $A - V$ in a fixed ball when the "Stummel norm of V " is small.

LEMMA 1.3. *Set $\Omega = B_2(0)$ and fix p , $n/2 < p < \infty$. There exist constants $\delta = \delta(\lambda, \eta)$ and $C = C(\lambda, p)$ such that if*

$$\sup_{x \in \Omega} \int_{\Omega} \frac{|V(y)|}{|x - y|^{n-2}} dy \leq \delta,$$

then for any (smooth) u with $u|_{\partial\Omega} = 0$ we have $\|u\|_{L^\infty(\Omega)} \leq C\|Au - Vu\|_{L^p(\Omega)}$.

PROOF. Let $G_{A,\Omega}(x, y)$ denote the Green's function for the operator A and domain Ω . Then, setting $f = Au - Vu$,

$$u(x) = - \int_{\Omega} G_{A,\Omega}(x, y) f(y) dy - \int_{\Omega} G_{A,\Omega}(x, y) V(y) u(y) dy.$$

Since $G_{A,\Omega}(x, y) \leq C/|x - y|^{n-2}$ ([7]),

$$\|u\|_{L^\infty(\Omega)} \leq C(\lambda, p) \|f\|_{L^p(\Omega)} + C(\lambda) \sup_{x \in \Omega} \int_{\Omega} \frac{|V(y)|}{|x - y|^{n-2}} dy \|u\|_{L^\infty(\Omega)}.$$

The conclusion of Lemma 1.3 is immediate.

We now begin with the principal results of the paper.

THEOREM 1.4. *There exist $r_0 = r_0(\eta, \lambda)$ and for each $p \in (0, \infty)$ a constant $C = C(p, \lambda)$ such that if u is a solution to $Au = Vu$ in Ω , then*

$$\sup_{B_{r/2}} |u| \leq C \left(\int_{B_r} |u|^p \right)^{1/p},$$

provided $0 < r \leq r_0$ and $B_{2r} \subset \Omega$.

PROOF. Again by translation we may assume the center of B_r is the origin. As before the function $u_r(x) \equiv u(rx)$ is a solution in $B_2 \equiv B_2(0)$ of $A_r u_r = V_r u_r$, where $A_v = \sum_{i,j=1}^n D_{x_i}(a_{ij}(rx) D_{x_j})$ and $V_r(x) = r^2 V(rx)$. Also the

$$\sup_{x \in B_2} r^2 \int_{B_2} \frac{|V(rx)|}{|x - y|^{n-2}} dy \leq \sup_{0 < \alpha < 8r} \sup_{w \in \Omega} \int_{B_\alpha(w)} \frac{|V(z)|}{|w - z|^{n-2}} dz.$$

This last expression tends to zero with r . Hence Theorem 1.4 is a consequence of the following result:

LEMMA 1.5. Let $\delta = \delta(\lambda, n)$ denote the number specified in Lemma 1.3. If

$$\sup_{x \in B_2} \int_{B_2} \frac{|V(y)|}{|x - y|^{n-2}} dy \leq \delta,$$

then a solution of $Au = Vu$ in B_4 satisfies

$$\|u\|_{L^\infty(B_{1/2})} \leq C(\lambda, n, p) \left(\int_{B_1} |u|^p dx \right)^{1/p}, \quad 0 < p < \infty.$$

PROOF. Let $G(x, y) \equiv G_{A-V, B_2}(x, y)$ denote the Green's function for $A - V$ and B_2 . (Existence of the Green's function is guaranteed by Lemma 1.3.) Pick numbers $1/2 \leq s < t \leq 1$ and a function $\varphi \in C_0^\infty(B_{t-(t-s)/4})$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_{(t+s)/2}$, and $|\nabla \varphi| \leq c/(t-s)$. Then

$$\begin{aligned} u(x)\varphi(x) &= \int_{B_2} \nabla_y G(x, y) \cdot a(\nabla \varphi)(y) u(y) dy \\ &\quad - \int_{B_2} G(x, y) a(\nabla u)(y) \cdot \nabla \varphi(y) dy. \end{aligned}$$

Hence

$$\begin{aligned} |u(x)\varphi(x)| &\leq \frac{c}{t-s} \left(\int_{B_{t-(t-s)/4} \setminus B_{(t+s)/2}} |\nabla_y G(x, y)|^2 dy \right)^{1/2} \left(\int_{B_t} u^2(y) dy \right)^{1/2} \\ &\quad + \frac{c}{t-s} \left(\int_{B_{t-(t-s)/4} \setminus B_{(t+s)/2}} G(x, y)^2 dy \right)^{1/2} \left(\int_{B_{t-(t-s)/4} \setminus B_{(t+s)/2}} |\nabla u(y)|^2 dy \right)^{1/2}. \end{aligned}$$

Using Lemmas 1.1 and 1.2 it is not difficult to see that

$$\left(\int_{B_{t-(t-s)/4} \setminus B_{(t+s)/2}} |\nabla u(y)|^2 dy \right)^{1/2} \leq \frac{c}{(t-s)^{3/2}} \left(\int_{B_t} u^2 \right)^{1/2},$$

while if $x \in B_s$,

$$\left(\int_{B_{t-(t-s)/4} \setminus B_{(t+s)/2}} |\nabla_y G(x, y)|^2 dy \right)^{1/2} \leq \frac{c}{(t-s)^{3/2}} \int_{B_t} G(x, y) dy$$

and

$$\left(\int_{B_{t-(t-s)/2} \setminus B_{(t+s)/2}} G(x, y)^2 dy \right)^{1/2} \leq \frac{c}{(t-s)^{1/2}} \int_{B_t} G(x, y) dy.$$

Lemma 1.3 implies the boundedness in x of $\int_{B_t} G(x, y) dy$, and so combining our results we have $\|u\|_{L^\infty(B_s)} \leq c/(t-s)^3 (\int_{B_t} u^2)^{1/2}$.

An inequality of the above type always implies the final result, i.e. for each $p \in (0, \infty)$, $\|u\|_{L^\infty(B_{1/2})} \leq C_p (\int_{B_1} |u|^p)^{1/p}$. The reader can find the proof of this fact in [5, pp. 1004–1005]. The proof is also an easy modification of the argument found in Lemma 1.2.

2. Properties of the Green's function and the infimum estimate. The object of this section is to prove the following

THEOREM 2.1. *Suppose $u > 0$ is a solution of $Au = Vu$ in Ω . There exist positive constants $r_0 = r_0(\lambda, \eta)$, $p_0 = p_0(\lambda)$, and $C = C(\lambda)$, all independent of u , such that*

$$\left(\int_{B_r} u^{p_0} \right)^{1/p_0} \leq C \inf_{B_{r/2}} u$$

provided $r \leq r_0$ and $B_{2r} \subset \Omega$.

The proof of the theorem relies on some lemmas each having some independent interest.

LEMMA 2.2. *Let $u > 0$ be a solution of $Au = Vu$ in Ω . There exists a constant $C = C(\lambda, \eta)$ such that, if $B_{2r} \subset \Omega$, then*

$$\int_{B_r} \left| \log u - \int_{B_r} \log u \right|^2 dx \leq C.$$

PROOF. Let $\varphi \in C_0^\infty(B_{3r/2})$, $\varphi \equiv 1$ on B_r , $0 \leq \varphi \leq 1$, $|\nabla \varphi| \leq C/r$. Then

$$\begin{aligned} \int |\nabla \log u|^2 \varphi^2 dx &= \int \frac{|\nabla u|^2}{u^2} \varphi^2 dx \leq \lambda^{-1} \int a \nabla u \cdot \nabla u \frac{\varphi^2}{u^2} dx \\ &\leq -\lambda^{-1} \int a \nabla u \cdot \nabla \left(\frac{\varphi^2}{u} \right) dx + 2\lambda^{-1} \int a \nabla u \cdot \nabla \varphi \frac{\varphi}{u} dx \\ &\leq \lambda^{-1} \int |V| \varphi^2 + 2\lambda^{-1} \int \nabla u \cdot \nabla \varphi \frac{\varphi}{u} dx. \end{aligned}$$

Since $V \in K_\eta$, if $B_{2r} \equiv B_{2r}(x)$, then

$$\int_{B_{2r}} |V| dy \leq 2^{n-2} r^{n-2} \int_{|x-y| < 2r} \frac{|V(y)|}{|x-y|^{n-2}} dy \leq Cr^{n-2},$$

where $C = C(\lambda, \eta)$.

This immediately gives the estimate

$$\int_{B_r} |\nabla \log u|^2 dx \leq Cr^{n-2},$$

and from this and Poincaré's inequality the conclusion of the lemma.

LEMMA 2.3. *Let $u > 0$ be a solution of $Au = Vu$ in Ω , and let r_0 be the number determined in Theorem 1.4. Then there exists a constant $C = C(\lambda, \eta)$ such that $\int_{B_{2r}} u dx \leq C \int_{B_r} u dx$ for $0 < r \leq r_0$ and $B_{4r} \subset \Omega$.*

PROOF. By Lemma 2.2 and the John-Nirenberg lemma [6], there exist positive numbers $\varepsilon = \varepsilon(\lambda, \eta)$ and $C = C(\lambda, \eta)$ such that

$$\left(\int_{B_r} u^\varepsilon dx \right) \left(\int_{B_r} u^{-\varepsilon} dx \right) \leq C$$

provided that $B_{4r} \subset \Omega$. As a consequence, see [2 or 9], we obtain, for every B_r such that $B_{4r} \subset \Omega$, $\int_{B_{2r}} u^\varepsilon dx \leq C \int_{B_r} u^\varepsilon dx$. Using Theorem 1.4, the above doubling property of u^ε and Hölder's inequality,

$$\int_{B_{2r}} u dx \leq C \left(\int_{B_{3r}} u^\varepsilon dx \right)^{1/\varepsilon} \leq C \left(\int_{B_r} u^\varepsilon dx \right)^{1/\varepsilon} \leq C \int_{B_r} u dx.$$

LEMMA 2.4. Set $V^+ = \max(V, 0)$ and let $G(x, y) \equiv G_{A-V^+, \Omega}(x, y)$ denote the Green's function corresponding to $A - V^+$ and Ω . Take r_0 as in Theorem 1.4. Then for any q , $1 < q < n/(n-2)$, there exists a constant $C = C(q, \lambda, \eta)$ such that

$$\left(\int_{B_r} G(x, y)^q dy \right)^{1/q} \leq C \int_{B_r} G(x, y) dy,$$

provided $B_{4r} \subset \Omega$ and $0 < r \leq r_0$.

PROOF. If $x \in \Omega \setminus B_{3r}$ then the result follows from Theorem 1.4 and Lemma 2.3. So assume $x \in B_{3r}$. Let $G_r(x, y) = G_{A-V^+, B_{4r}}(x, y)$. By the maximum principle $G_r \leq G$. Then

$$\int_{B_r} G(x, y)^q dy \leq C \int_{B_r} (G(x, y) - G_r(x, y))^q dy + C \int_{B_r} G_r(x, y)^q dy.$$

Since $G(x, \cdot) - G_r(x, \cdot)$ is a nonnegative solution of $(A - V^+)u = 0$ in B_{4r} , again by Theorem 1.4 and Lemma 2.3 we have

$$\int_{B_r} (G(x, y) - G_r(x, y))^q dy \leq C \left(\int_{B_r} (G(x, y) - G_r(x, y)) dy \right)^q.$$

To handle the term $\int_{B_r} G_r(x, y)^q dy$ we assume that the center of B_r is the origin. Then

$$\begin{aligned} \int_{B_r} G_r(x, y)^q dy &= c \int_{B_1} G_r(rx, ry)^q dy \\ &= c \left(\int_{B_1} G_{A_r-V_r^+, B_4}(x, y)^q dy \right) r^{(n-2)q}, \end{aligned}$$

where, as before, $A_r = \sum_{i,j=1}^n D_i(a_{ij}(rx)D_j)$ and $V_r^+(x) = r^2 V^+(rx)$. Since

$$\int_{B_r} G_r(x, y) dy = c \left(\int_{B_1} G_{A_r-V_r^+, B_4}(x, y) dy \right) r^{n-2},$$

the proof of Lemma 2.4 will be complete once we prove the following

CLAIM. Let $Au - Vu = \chi_{B_1}$ in B_4 , $u|_{\partial B_4} = 0$. (χ_E = characteristic function of E .) If

$$\sup_{B_4} \int_{B_4} \frac{|V(y)|}{|x-y|^{n-2}} dy \leq \delta(\lambda),$$

then there exists a $C = C(\lambda) > 0$ such that $\inf_{B_3} u \geq C$.

We need only observe

$$u(x) = \int_{B_1} G_{A,B_4}(x, y) dy - \int_{B_1} G_{A,B_4}(x, y) V(y) u(y) dy,$$

the

$$\inf_{x \in B_3} \int_{B_1} G_{A,B_4}(x, y) dy \geq c(\lambda, n) > 0$$

while

$$\left| \int_{B_1} G_{A,B_4}(x, y) V(y) u(y) dy \right| \leq \delta(\lambda) \|u\|_{L^\infty(B_1)}.$$

From Lemma (1.3), $\|u\|_{L^\infty(B_1)}$ is bounded by a constant depending only on dimension and λ . Hence the claim is justified.

We are now ready to give the

PROOF OF THEOREM 2.1. Take r_0 and δ as in Theorem 1.4. By translation and dilation we reduce the estimate to the case $\Omega = B_2 = B_2(0)$, B_1 , $B_{1/2}$ also centered at the origin, and

$$\sup_{x \in B_2} \int_{B_2} \frac{|V(y)|}{|x - y|^{n-2}} dy \leq \delta(\lambda),$$

a fixed small positive number. We want to show that if u is a nonnegative solution of $Au = Vu$ in B_2 , then u satisfies $\inf_{B_{1/2}} u \geq c(\int_{B_1} u^{p_0})^{1/p_0}$ for some $0 < p_0 = p_0(\lambda)$ and $c = c(\lambda)$.

We first show if $u \geq 1$ on a closed set $\Gamma \subset B_1$, then $\inf_{B_{1/2}} u \geq c|\Gamma|^M$, where c and M depend only on λ and η . In fact, in this case, since $Au - V^+u = -V^-u \leq 0$, the maximum principle implies the existence of $c = c(\lambda, \eta)$ such that

$$u(x) \geq c \int_{\Gamma} G_{A-V^+}(x, y) dy.$$

The reverse Hölder inequality (Lemma 2.4) implies

$$\int_{\Gamma} G_{A-V^+}(x, y) dy \geq c|\Gamma|^M \int_{B_{3/2}} G_{A-V^+}(x, y) dy,$$

where c and M depend only on λ and η (see [2 or 9]). The “claim” in Lemma 2.4 implies

$$\inf_{x \in B_{1/2}} \int_{B_{3/2}} G_{A-V^+}(x, y) dy \geq c(\lambda) > 0.$$

Hence $\inf_{B_{1/2}} u \geq c|\Gamma|^M$.

The above inequality gives an estimate for the distribution function of a general nonnegative solution u of $Au = Vu$. In fact, if $\Gamma = \{x \in B_1 | u(x) \geq t\}$, then $\inf_{B_{1/2}} u \geq ct|\Gamma|^M$. From this it is easy to conclude that, for $0 < p_0 < 1/M$,

$$\left(\int_{B_1} u^{p_0} \right)^{1/p_0} \leq C \inf_{B_{1/2}} u.$$

The immediate consequence of Theorems 1.4 and 2.1 is the (uniform) Harnack inequality, namely

THEOREM 2.5. *There exist constants r_0 and C depending only on λ and η such that any nonnegative solution of $Au = Vu$ in Ω satisfies $\sup_{B_{r/2}} u \leq C \inf_{B_{r/2}} u$ provided $0 < r \leq r_0$ and $B_{2r} \subset \Omega$.*

3. Continuity. In this section we will determine a modulus of continuity valid for all solutions of $Au - Vu = 0$ in Ω .

THEOREM 3.1. *There exist positive numbers $\alpha = \alpha(\lambda)$ and $C = C(\lambda)$ such that for any solution u of $Au = Vu$ in Ω , any ball $B_r(x_0)$ with $B_{4r}(x_0) \subset \Omega$, and any $x \in B_r(x_0)$, we have*

$$|u(x) - u(x_0)| \leq C \left[\left(\frac{|x - x_0|}{r} \right)^{\alpha/2} (1 + \eta(2r)) + \eta(2r^{1/2}|x - x_0|^{1/2}) \right] \sup_{B_{3r}(x_0)} |u|.$$

PROOF. Given $B_r(x_0)$, pick $\varphi \in C_0^\infty(B_{2r}(x_0))$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_{3r/2}(x_0)$, and $|\nabla \varphi| \leq c/r$. Let $\Gamma(x, y)$ denote the fundamental solution of the operator A . Then

$$\begin{aligned} u(x)\varphi(x) &= - \int \Gamma(x, y)V(y)\varphi(y)u(y) dy \\ &\quad + \int \Gamma(x, y)a(y)(\nabla u(y)) \cdot \nabla \varphi(y) dy \\ &\quad - \int \nabla_y \Gamma(x, y) \cdot a(y)(\nabla \varphi(y))u(y) dy. \end{aligned}$$

For $x \in B_r(x_0)$ we have

$$\begin{aligned} u(x) - u(x_0) &= - \int (\Gamma(x, y) - \Gamma(x_0, y))V(y)\varphi(y)u(y) dy \\ &\quad + \int (\Gamma(x, y) - \Gamma(x_0, y))a(y)(\nabla u(y)) \cdot \nabla \varphi(y) dy \\ &\quad - \int (\nabla_y \Gamma(x, y) - \nabla_y \Gamma(x_0, y)) \cdot a(y)(\nabla \varphi(y))u(y) dy \\ &\equiv -\text{I} + \text{II} - \text{III}. \end{aligned}$$

We begin by estimating II and III. From known Hölder estimates on solutions of $Au = 0$ [4, 8, 10] we have the existence of $\alpha = \alpha(\lambda) > 0$ such that if

$$y \in B_{2r}(x_0) \setminus B_{3r/2}(x_0)$$

then

$$|\Gamma(x, y) - \Gamma(x_0, y)| \leq C(\lambda) \left(\frac{|x - x_0|}{r} \right)^\alpha \Gamma(x_0, y).$$

Since $\Gamma(x_0, y) \leq C(\lambda)/|x_0 - y|^{n-2}$ we have

$$|\text{II}| \leq C(\lambda) \left(\frac{|x - x_0|}{r} \right)^\alpha r \left(\int_{B_{2r}} |\nabla u|^2 dx \right)^{1/2}.$$

Finally Lemma 1.1 implies

$$|\text{II}| \leq C(\lambda) \left(\frac{|x - x_0|}{r} \right)^\alpha \left(\int_{B_{3r}} u^2 \right)^{1/2}.$$

To handle III we first use Schwartz's inequality to obtain

$$\begin{aligned} |\text{III}| &\leq \frac{C(\lambda)}{r} \left(\int_{B_{2r}(x_0) \setminus B_{3r/2}(x_0)} |\nabla_y \Gamma(x, y) - \nabla_y \Gamma(x_0, y)|^2 dy \right)^{1/2} \\ &\quad \times \left(\int_{B_{2r}(x_0)} u^2 dy \right)^{1/2}. \end{aligned}$$

From the usual Caccioppoli inequality [7, p. 51]

$$\begin{aligned} &\left(\int_{B_{2r}(x_0) \setminus B_{(3/2)r}(x_0)} |\nabla_y \Gamma(x, y) - \nabla_y \Gamma(x_0, y)|^2 dy \right)^{1/2} \\ &\leq \frac{C(\lambda)}{r} \left(\int_{B_{3r}(x_0) \setminus B_{(5/4)r}(x_0)} |\Gamma(x, y) - \Gamma(x_0, y)|^2 dy \right)^{1/2}. \end{aligned}$$

This last expression is once again bounded by $C(\lambda)/r^{n/2-1}(|x - x_0|/r)^\alpha$. Hence

$$|\text{III}| \leq C(\lambda) \left(\frac{|x - x_0|}{r} \right)^\alpha \left(\int_{B_{2r}} u^2 \right)^{1/2}.$$

Finally, to bound the first term I we first write it as

$$\begin{aligned} I &= \int_{|x_0 - y| > N|x - x_0|} (\Gamma(x, y) - \Gamma(x_0, y)) V(y) u(y) \varphi(y) dy \\ &\quad + \int_{|x_0 - y| \leq N|x - x_0|} (\Gamma(x, y) - \Gamma(x_0, y)) V(y) u(y) \varphi(y) dy, \end{aligned}$$

where N is a large positive number to be appropriately chosen later. For the first integral on the right side we have

$$|\Gamma(x, y) - \Gamma(x_0, y)| \leq \frac{C(\lambda)}{N^\alpha} \Gamma(x_0, y) \leq \frac{C(\lambda)}{N^\alpha} \frac{1}{|x_0 - y|^{n-2}}.$$

In the second integral we only use the pointwise bound on $\Gamma(x, y)$, namely $\Gamma(x, y) \leq C(\lambda)/|x - y|^{n-2}$. We easily conclude that

$$\begin{aligned} |\text{I}| &\leq \frac{C(\lambda)}{N^\alpha} \int_{|x_0 - y| < 2r} \frac{|V(y)|}{|x_0 - y|^{n-2}} dy \sup_{B_{2r}(x_0)} |u| \\ &\quad + C(\lambda) \eta((N+1)|x_0 - x|) \sup_{B_{2r}(x_0)} |u|. \end{aligned}$$

Choose $N = (r/|x - x_0|)^{1/2}$. Then

$$|\text{I}| \leq c(\lambda) \left[\left(\frac{|x - x_0|}{r} \right)^{\alpha/2} \eta(2r) + \eta(2r^{1/2}|x_0 - x|^{1/2}) \right] \sup_{B_{2r}(x_0)} |u|.$$

Our estimates for the terms I, II, and III give the conclusion of Theorem 3.1.

Recently Professor Umberto Mosco kindly gave us a preprint of his work with Gianni Dal Maso which also contains a continuity result for solutions of $Au = Vu$ in Ω [3].

REFERENCES

1. M. Aizenman and B. Simon, *Brownian motion and Harnack's inequality for Schrödinger operators*, Comm. Pure Appl. Math. **35** (1982), 209–271.
2. R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.
3. G. Dal Maso and U. Mosco, *Weiner criteria and energy decay for relaxed Dirichlet problems*, preprint.
4. E. De Giorgi, *Sulla differenziabilità e l'analiticità degli integrali multipli regolari*, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (3) **3** (1957), 25–43.
5. E. B. Fabes and D. W. Stroock, *The L^p -integrability of Green's functions and fundamental solutions for elliptic and parabolic equations*, Duke Math. J. **51** (1984), 997–1016.
6. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 577–591.
7. W. Littman, G. Stampacchia and H. F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 45–79.
8. J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591.
9. B. Muckenhoupt, *The equivalence of two conditions for weight functions*, Studia Math. **49** (1974), 101–106.
10. J. Nash, *Continuity of the solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954.
11. M. V. Safanov, *Harnack's inequality for elliptic equations and the Hölder property of their solutions*, J. Soviet Math. **21** (1983), 851–863.
12. M. Schechter, *Spectra of partial differential operators*, North-Holland, 1971.
13. N. S. Trudinger, *Local estimates for subsolutions and supersolutions of general second order elliptic quasilinear equations*, Invent. Math. **61** (1980), 67–79.
14. Z. Zhao, *Conditional gauge with unbounded potential*, Z. Wahrsch. Verw. Gebiete **65** (1983), 13–18.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CATANIA, CATANIA, ITALY

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BOLOGNA, BOLOGNA, ITALY

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455