ON THE HOMOTOPY GROUPS OF $A(X)$

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ABSTRACT. In this paper we will prove that if X is any space with a finite fundamental group, then Waldhausen's algebraic K -groups of X are finitely generated. We will use Dwyer's machinery developed in *Twisted homological stability for general linear groups* (Ann. of Math. 111).

Introduction. In [D] Dwyer proved the following theorem:

THEOREM (DWYER). *If X is simply-connected and $\pi_i(X)$ is finitely generated for $i \geq 2$, then $\pi_j(A(X))$ is finitely generated for all j , where $A(X)$ is Waldhausen's algebraic K -theory of a space X .*

In this paper we will use Dwyer's approach in order to generalize his theorem to the case when $\pi_1(X)$ is a finite group. We will prove the following theorem.

THEOREM I. *If $\pi_1(X)$ is a finite group and $\pi_i(X)$ is finitely generated for $i \geq 2$, then $\pi_j(A(X))$ is finitely generated for all j .*

REMARK. Theorem I is, in general, the best possible result in this direction. Consider $X = S^1$. Then using a spectral sequence $E_{p,q}^2 \Rightarrow \pi_{p+q}(A(X))$ (for details see [W, Proposition 2.6]) where

(i) $E_{p,q}^2 = \pi_{p+q}(\text{fiber}(A^{q-1}(X) \rightarrow A^{q-2}(X)))$ if $q \geq 2$,

(ii) $E_{p,q}^2 = \pi_{p+1}K(Z[G(X)])$ if $q = 1$

and where $\cdots \rightarrow A^j(X) \rightarrow \cdots$ is a tower derived from the Postnikov tower for $\Omega^\infty S^\infty$, it is easy to see that $\pi_2(A(S^1))$ contains an infinite number of copies of $Z/2Z$ so $\pi_2(A(S^1))$ is not finitely generated.

I. Let R be a ring and let A be an R -bimodule which is finitely generated as an abelian group. Then $M_n A$ —the abelian group of $(n \times n)$ -matrices with entries in A —is an $M_n R$ -bimodule and $M_n A$ becomes a left $\text{Gl}_n R$ -module by defining

$$(1.1) \quad g \circ m = gmg^{-1} \quad \text{for } m \in M_n A, g \in \text{Gl}_n R \text{ where the multiplication}$$

is taken with respect to the R -bimodule structure on A .

Now let A^n be a left $\text{Gl}_n R$ -module with the obvious action. Consider $\bar{R}^n$ —the left $\text{Gl}_n R$ -module which is isomorphic to R^n as an abelian group and where the action of $\text{Gl}_n R$ is given by

$$(1.2) \quad gx(r_1, \dots, r_n) = (r_1, \dots, r_n)g^{-1} \quad \text{for all } g \in \text{Gl}_n R.$$

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The abelian group $\overline{R}^n$ is also a left R -module with the obvious action and these two left module structures on $\overline{R}^n$ commute:

$$\begin{aligned} gx(r_1, \dots, r_n) &= gx(rr_1, \dots, rr_n) = (rr_1, \dots, rr_n)g^{-1} \\ &= r((r_1, \dots, r_n)g^{-1}) = r(gx(r_1, \dots, r_n)) \end{aligned}$$

for $g \in \text{Gl}_n R$, $r \in R$, $(r_1, \dots, r_n) \in \overline{R}^n$.

PROPOSITION 1.3. *The $\text{Gl}_n R$ -module $M_n A$ given by (1.1) is isomorphic to $A^n \otimes_R \overline{R}^n$, where $A^n \otimes_R \overline{R}^n$ is a left $\text{Gl}_n R$ -module by the diagonal action.*

PROOF. We have the map $F: A^n \otimes_R \overline{R}^n \rightarrow M_n A$ given by

$$(a_1, \dots, a_n) \otimes (r_1, \dots, r_n) \xrightarrow{F} \begin{vmatrix} a_1 r_1, \dots, a_1 r_n \\ \vdots & \vdots \\ a_n r_1, \dots, a_n r_n \end{vmatrix}.$$

This is obviously a group isomorphism. It is easy to see by computing both sides that for any $g \in \text{Gl}_n R$, $(a_1, \dots, a_n) \in A^n$, and $(r_1, \dots, r_n) \in \overline{R}^n$ we have

$$g \circ F((a_1, \dots, a_n) \otimes (r_1, \dots, r_n)) = F(g(a_1, \dots, a_n) \otimes gx(r_1, \dots, r_n)),$$

so F is a $\text{Gl}_n R$ isomorphism of modules.

In the following proposition we will use van der Kallen's definitions and notation from [K] (see also [D]):

PROPOSITION 1.4. *Let $T: (\text{Gl}_n R\text{-left modules})^2 \rightarrow \text{Ab}$ be a functor of finite degree. Then $T(A^n, \overline{R}^n)$ is a strongly stable system for $\text{Gl}_n R$.*

PROOF. Do precisely the same as in the proof of Dwyer's Lemma 3.1 in [D].

We can obtain immediately the following corollary from Proposition 1.4.

COROLLARY 1.5. *In the notation from (1.1) and (1.3), $M_n A$ is a strongly stable system of coefficients for $\text{Gl}_n R$.*

II. Now we are able to prove Theorem I. Let us recall (see [W, §2] for details) that $A(X)$ is an H -space with the identity component given by $\widehat{B\text{Gl}}(X)^+$, so Theorem I follows easily from the following lemma (see [D, Lemma 4.1]):

LEMMA 2.1. *If X is as in Theorem I, then the integral homology groups of $\widehat{B\text{Gl}}(X)$ are finitely generated.*

Before we prove Lemma 2.1 let us make the following remark.

REMARK 2.2. The properties of $\widehat{B\text{Gl}}(X)$ which we need are

- (i) $\pi_1(\widehat{B\text{Gl}}(X)) = \text{Gl}(Z\pi)$, where $\pi = \pi_1(X)$;
- (ii) $\pi_k(\widehat{B\text{Gl}}(X))$ is isomorphic as a $\text{Gl}(Z\pi)$ -module to

$$M(\pi_{k-1}^s(\Omega X \cup \text{pt})) = \varinjlim_n M_n(\pi_{k-1}^s(\Omega X \cup \text{pt}))$$

with the action given by (1.1).

Let $\mathfrak{M}$ be the smallest class of $\text{Gl}(Z\pi)$ -modules which has the following properties:

- (1) $\mathfrak{M}$ contains $M(A)$ for all $Z\pi$ -bimodules A , which are finitely generated as abelian groups;

- (2) $\mathfrak{M}$ contains all finitely generated abelian groups with the trivial π -action;
 (3) If $\mathfrak{M}$ contains A and B , then $\mathfrak{M}$ contains $H_j(K(A, n); B)$ for $n \geq 2$ and $j \geq 0$.

We will need the following lemma.

LEMMA 2.3. *If $A \in \mathfrak{M}$, then the groups $H_j(\mathrm{Gl}(Z\pi); A)$ are finitely generated.*

PROOF OF LEMMA 2.1 FROM LEMMA 2.3. Let $P_k \widehat{\mathrm{BGl}}(X)$ be a k th stage of the Postnikov system for $\widehat{\mathrm{BGl}}(X)$. We will show by induction on k that for any integer k and $A \in \mathfrak{M}$ the homology groups $H_j(P_k \widehat{\mathrm{BGl}}(X); A)$ are finitely generated.

(i) If $k = 1$, then this is exactly Lemma 2.3.

(ii) Let $k > 1$. Look at the fibration

$$K(\pi_k(\widehat{\mathrm{BGl}}(X)), k) \rightarrow P_k(\widehat{\mathrm{BGl}}(X)) \rightarrow P_{k-1}(\widehat{\mathrm{BGl}}(X)).$$

Then for any $A \in \mathfrak{M}$ we have the Serre spectral sequence with twisted coefficients:

$$E_{p,q}^2 = H_p(P_{k-1}(\widehat{\mathrm{BGl}}(X)); H_q(K(\pi_k(\widehat{\mathrm{BGl}}(X)), k); A))$$

converging to $H_{p+q}(P_k(\widehat{\mathrm{BGl}}(X)); A)$.

But $\pi_k(\widehat{\mathrm{BGl}}(X)) = M(\pi_{k-1}^s(\Omega X \cup \mathrm{pt})) \in \mathfrak{M}$, so $H_q(K(\pi_k(\widehat{\mathrm{BGl}}(X)), k); A) \in \mathfrak{M}$ and by the induction hypothesis all groups in E_{**}^2 are finitely generated. This gives us immediately that for all j , $H_j(P_k(\widehat{\mathrm{BGl}}(X)); A)$ are finitely generated.

Now we will need two more lemmas (see [D, Lemmas 4.3 and 3.2]).

LEMMA 2.4. *The functor $T: \mathrm{Ab}^2 \rightarrow \mathrm{Ab}$ given by $T(A, B) = H_j(K(A, n); B)$, $n \geq 2$, $j \geq 0$, is of finite degree with respect to A and B .*

PROOF. Precisely the same as the proof of Dwyer's Lemma 4.3.

LEMMA 2.5. *Let A, B, C be abelian categories and let $S: A \rightarrow B$ and $T: B \rightarrow C$ be functors. If S has degree k and T has degree l , then the composite TS has degree $\leq kl$.*

Now we are able to prove Lemma 2.3 which will finish the proof of Theorem I. We will need two simple observations:

2.6. If π is a finite group, then $\mathrm{Gl}_n(Z\pi)$ is an arithmetic group. This is the special case of the much more general situation (see [S, Example 5]): if R is a ring which is finitely generated as a Z -module, then R^* is an arithmetic group. In our case take $R = M_n(Z\pi)$.

2.7. If π is a finite group, then van der Kallen's stability theorems with twisted coefficients hold for $Z\pi$ (see [K] for details).

Every module in the class $\mathfrak{M}$ can be expressed as $T(M(A))$, where A is finitely generated over Z and T is a functor of finite degree which commutes with direct limits and preserves the subcategory of finitely generated abelian groups. Then

$$T(M(A)) = \varinjlim_n T(M_n(A))$$

and

$$H_j(\mathrm{Gl}(Z\pi); T(M(A))) = \varinjlim_n H_j(\mathrm{Gl}_n(Z\pi); T(M_n(A))).$$

By Proposition 1.4 and Lemma 2.5, $T(M_n(A))$ is a strongly stable system for $\mathrm{Gl}_n(Z\pi)$ so

$$\varinjlim_n H_j(\mathrm{Gl}_n(Z\pi); T(M_n(A))) = H_j(\mathrm{Gl}_N(Z\pi); T(M_N(A)))$$

for sufficiently large N (by (2.7)). In [R] Raghunathan proved that for an arithmetic group G and a ZG -module M which is finitely generated as an abelian group, $H_j(G; M)$ is finitely generated for all $j \geq 0$. So by observation (2.6) and [R] we know that the homology groups $H_j(\mathrm{Gl}_N(Z\pi); T(M_N(A)))$ are finitely generated. This completes the proof of Lemma 2.3.

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