MONOTONE AND OPEN WHITNEY MAPS

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ABSTRACT. In this paper we prove that if X is a locally connected continuum, then there exists a monotone and open Whitney map for 2^X .

1. Introduction. A *continuum* is a nonempty compact connected metric space. A *Peano continuum* is a locally connected continuum. The *hyperspaces* of a continuum X are the spaces $2^X = \{A \subset X: A \text{ is nonempty and compact}\}$ and $C(X) = \{A \in 2^X: A \text{ is connected}\}$ metrized with the Hausdorff metric. A *Whitney map* for the hyperspace $\mathcal{H}$ of X is a continuous function $\omega: \mathcal{H} \rightarrow [0, \infty)$ such that $\omega(\{x\}) = 0$ for each $x \in X$, and if $A, B \in \mathcal{H}$ and $A \subset B \neq A$, then $\omega(A) < \omega(B)$ is *monotone* if $\omega^{-1}(t)$ is connected for every $t \in [0, \infty)$.

It is known that every Whitney map for $C(X)$ is monotone and open [3, p. 1032]. In [6] it was shown that even when X is a circle or an arc, Whitney maps for 2^X can be constructed which are neither monotone nor open. S. B. Nadler, Jr. has asked in [6, 14.63 and 14.64] whether for every continuum X there exists a monotone (or an open) Whitney map for 2^X . Recently, W. J. Charatonik [2] answered both questions negatively by showing an example of a continuum X in the plane which has neither monotone nor open Whitney maps for 2^X . In this paper, we prove that if X is a Peano continuum, then a monotone open Whitney map can be constructed for 2^X .

Throughout this paper X will denote a continuum with a metric d . A metric ρ for X is said to be *convex* provided that, given $x, y \in X$, there exists $z \in X$ such that $\rho(x, z) = \rho(x, y)/2 = \rho(y, z)$. It is easy to prove that if ρ is a convex metric for X , then, for any $x, y \in X$, there exists a isometry $\sigma: [0, \rho(x, y)] \rightarrow X$ such that $\sigma(0) = x$ and $\sigma(\rho(x, y)) = y$. X is said to *admit a convex metric* ρ provided that ρ is a convex metric for X and the (original) topology on X is the same as the topology for X obtained by ρ . If X admits a convex metric ρ , then X is a Peano continuum [4]. The converse was proved independently by R. H. Bing [1] and E. E. Moise [5].

2. A monotone open Whitney map. Given a positive integer n and $A \in 2^X$, we define $\omega_n(A) = \inf\{\varepsilon > 0: \text{there exist } x_1, \dots, x_n \in X \text{ such that } A \subset B_\varepsilon(x_1) \cup \dots \cup B_\varepsilon(x_n)\}$, where $B_\varepsilon(x) = \{y \in X: d(x, y) < \varepsilon\}$. Then $\omega_n: 2^X \rightarrow [0, \infty)$ is continuous, $\omega_n(\{x\}) = 0$ for each $x \in X$ and $\omega_n(A) \leq \text{diameter of } X$. So, if we define $\omega: 2^X \rightarrow [0, \infty)$ by $\omega(A) = \sum \omega_n(A)/2^n$, then ω is continuous and $\omega(\{x\}) = 0$ for each $x \in X$.

To prove that ω is a Whitney map, we take $A, B \in 2^X$ such that $A \subset B \neq A$. Let b be a point of $B - A$ and let $\varepsilon > 0$ such that $B_{2\varepsilon}(b) \cap A = \emptyset$. Let m be

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the minimum positive integer for which there exist points $x_1, \dots, x_m$ of X such that $A \subset B_\varepsilon(x_1) \cup \dots \cup B_\varepsilon(x_m)$. Then there exists $\alpha > 0$ such that $\alpha < \varepsilon$ and $A \subset B_\alpha(x_1) \cup \dots \cup B_\alpha(x_m)$. So $\omega_m(A) \leq \alpha < \varepsilon$. The choice of ε and m implies that there are no points $y_1, \dots, y_m \in X$ such that $B \subset B_\varepsilon(y_1) \cup \dots \cup B_\varepsilon(y_m)$. Then $\omega_m(B) \geq \varepsilon > \omega_m(A)$. This implies that $\omega(B) > \omega(A)$. Hence ω is a Whitney map.

THEOREM. *If d is a convex metric for X , then the Whitney map defined above is monotone and open.*

PROOF. Given a finite subset $F = \{p_1, \dots, p_n\}$ of X , $x_0 \in X$ and $\beta > 0$ such that $F \subset B_\beta(x_0)$, we put $t_1 = d(x_0, p_1), \dots, t_n = d(x_0, p_n)$ and we choose isometries $\sigma_i: [0, t_i] \rightarrow X$ such that $\sigma_i(0) = x_0$ and $\sigma_i(t_i) = p_i$. We define a path $F: [0, 1] \rightarrow 2^X$ by $F(t) = \{\sigma_1(tt_1), \dots, \sigma_n(tt_n)\}$. Then, if $t \in [0, 1]$, $F(t) \subset B_{t\beta}(x_0)$, so that $\omega_1(F(t)) \leq t\beta$. It is easy to see that $\omega_m(F(t)) \leq \omega_m(F) + (1-t)\beta$ for each m , and that for each $m \geq n$, $\omega_m(F(t)) = 0 = \omega_m(F)$. This implies that $\omega(F(t)) \leq t\beta/2 + \omega(F) - \omega_1(F)/2 + (1-t)\beta/2 - (1-t)\beta/2^{n-1} = \omega(F) - \omega_1(F)/2 + \beta/2 - (1-t)\beta/2^{n-1}$.

First, we will prove that ω is open. By Theorem 14.65 of [6], it is enough to prove that if we take a finite nondegenerate subset F of X and if we take an open subset U of 2^X such that $F \in U$, then $\omega(F)$ is an interior point of $\omega(U)$. Choose $0 < \varepsilon < 1$ such that $\{A \in 2^X: H(A, F) < \varepsilon \text{ (diameter of } X)\} \subset U$, where H is the Hausdorff metric for 2^X . We take $\beta < \omega_1(F)/(1 - \varepsilon/2^{n-1})$ and $x_0 \in X$ such that $F \subset B_\beta(x_0)$. Let us consider $F: [0, 1] \rightarrow 2^X$ constructed as above. Then $\omega(F(1 - \varepsilon/2)) \leq \omega(F) - \omega_1(F)/2 + \beta/2 - \varepsilon\beta/2^n < \omega(F)$. Moreover, given $t \in [1 - \varepsilon/2, 1]$, $H(F(t), F) < \varepsilon$ (diameter of X), so that $F([1 - \varepsilon/2, 1]) \subset U$.

It follows that there exists $\alpha > 0$ such that $(\omega(F) - \alpha, \omega(F)] \subset \omega(U)$. Now, we consider the continuum $M = \text{Cl}_X(B_{\varepsilon/2}(p_1))$. Then the set $M_1 = \{F \cup \{x\}: x \in M\}$ is connected, $M_1 \neq \{F\}$, $\omega(A) \geq \omega(F)$ for each $A \in M_1$ and $M_1 \subset U$. Then there exists $\lambda > 0$ such that the interval $[\omega(F), \omega(F) + \lambda)$ is contained in $\omega(U)$. Hence $\omega(F)$ is an interior point of $\omega(U)$. This proves that ω is open.

Now, we will prove that ω is monotone (compare with 14.66 of [6]). First, we will show that $\omega^{-1}([0, t])$ is connected for any $t \in \mathbf{R}$, $t > 0$. Since $\mathcal{E} = \{F \in 2^X: \omega(F) < t \text{ and } F \text{ is finite}\}$ is a dense subset of $\omega^{-1}([0, t])$, it will be enough to prove that $\mathcal{E}$ is connected. Take $F \in \mathcal{E}$. Let $\beta < \omega_1(F) + 2(t - \omega(F))$ and $x_0 \in X$ be such that $F \subset B_\beta(x_0)$. Then if we consider $F: [0, 1] \rightarrow 2^X$ as above, $\omega(F(s)) < t$ for each $s \in [0, 1]$. Moreover, $F(0) \in F_1(X) (= \{\{x\} \in 2^X: x \in X\})$, $F_1(X) \subset \mathcal{E}$ and $F_1(X)$ is connected. This implies that $\mathcal{E}$ is connected. Hence $\omega^{-1}([0, t])$ is connected for each $t \in \mathbf{R}$. It follows that $\omega^{-1}([0, t])$ is connected for each $t \in \mathbf{R}$.

Since 2^X is unicoherent [6, 1.176], to prove that $\omega^{-1}(t)$ is connected, it is enough to prove that $\omega^{-1}([t, \infty))$ is connected. Take $t \in \mathbf{R}$, and $A \in \omega^{-1}([t, \infty))$. Notice that $\{A \cup \{x\} \in 2^X: x \in X\}$ is a connected subset of $\omega^{-1}([t, \infty))$. Furthermore, $\{A \cup F \in 2^X: F \text{ is a finite subset of } X\}$ is a connected subset of $\omega^{-1}([t, \infty))$ containing X in its closure. This implies that $\omega^{-1}([t, \infty))$ is connected. Therefore, $\omega^{-1}(t)$ is connected, and ω is monotone.

COROLLARY. *If X is a Peano continuum, then there exists a monotone open Whitney map for 2^X .*

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