POWER SERIES AND SMOOTH FUNCTIONS EQUIVALENT TO A POLYNOMIAL

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ABSTRACT. An algebraic criterion is given for a power series in n variables over a field of characteristic 0 to be equivalent to a polynomial in $n - k$ variables over the ring of power series in k variables. For convergent power series over the reals or complexes a geometric interpretation of the criterion is established. An analogous sufficient condition is obtained for germs of smooth functions. Most of the previously known results follow easily from the criterion.

1. Introduction. Let $\mathbf{K}$ be a field of characteristic 0. We denote by $F_n = \mathbf{K}[[x_1, \dots, x_n]]$ the ring of formal power series in $x_1, \dots, x_n$ over $\mathbf{K}$. If $\mathbf{K}$ is equipped with an absolute value (in that case we always assume that $\mathbf{K}$ is complete), then $O_n = \mathbf{K}\{x_1, \dots, x_n\}$ denotes the subring of F_n of convergent power series. Let A_n be either of the rings F_n or O_n . Two power series f and g in A_n are said to be equivalent if there exists a $\mathbf{K}$ -automorphism of A_n transforming f onto g . Our goal in this note is to give criteria for a power series to be equivalent to a power series which is a polynomial with respect to at least some of the variables.

Given a p -tuple $\varphi = (\varphi_1, \dots, \varphi_p)$ of elements of A_n , we denote by $I(\varphi)$ the ideal of A_n generated by all $p \times p$ minors of the $p \times (p + n)$ matrix

$$M(\varphi) = \begin{pmatrix} \varphi_1 & 0 & \cdots & 0 & \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ 0 & \varphi_2 & \cdots & 0 & \frac{\partial \varphi_2}{\partial x_1} & \cdots & \frac{\partial \varphi_2}{\partial x_n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \varphi_p & \frac{\partial \varphi_p}{\partial x_1} & \cdots & \frac{\partial \varphi_p}{\partial x_n} \end{pmatrix}$$

Note that $I(u_1\varphi_1, \dots, u_p\varphi_p) = I(\varphi_1, \dots, \varphi_p)$ if $u_1, \dots, u_p$ are units in A_n . It is well known that A_n is a factorial ring and one easily shows that every element f in the maximal ideal $m(A_n)$ of A_n can be written as $f = \varepsilon f_1^{\lambda_1} \cdots f_p^{\lambda_p}$, where ε belongs to $\mathbf{K}^* = \mathbf{K} \setminus \{0\}$, $f_1, \dots, f_p$ are irreducible, relatively prime elements of A_n and $\lambda_1, \dots, \lambda_p$ are positive integers. Clearly, the ideal $J(f) = I(f_1, \dots, f_p)$ does not depend on the choice of $f_1, \dots, f_p$.

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For $k = 1, \dots, n$, we consider $F_k = \mathbf{K}[[x_1, \dots, x_k]]$ and $O_k = \mathbf{K}\{x_1, \dots, x_k\}$ as subrings of F_n and O_n , respectively.

THEOREM 1.1. *Let f be in $m(A_n)$. If the height of the ideal $J(f)$ is equal to $n - k$, then f is equivalent to a polynomial in $x_{k+1}, \dots, x_n$ with coefficients in A_k .*

To obtain interesting corollaries we note the following.

LEMMA 1.2. *For any element f in $m(A_n)$ the height of the ideal $J(f)$ is greater than or equal to 2.*

PROOF. Let $f_1, \dots, f_p$ be irreducible, relatively prime factors of f and let $g = f_1 \cdots f_p$. One checks easily that $J(f) = I(f_1, \dots, f_p)$ contains the ideal $I(g)$. By linearly changing variables, if necessary, and applying the Weierstrass preparation theorem, we may assume that $g = uP$, where u is a unit in A_n and P is a distinguished polynomial in x_n with coefficients in A_{n-1} . Clearly, $I(g) = I(P)$ and since P and $\partial P / \partial x_n$ have no common factor, the height of $I(P)$ is greater than or equal to 2.

COROLLARY 1.3. *Every power series in A_n , $n \geq 2$, is equivalent to a polynomial in x_{n-1}, x_n with coefficients in A_{n-2} . In particular, every element in A_2 is equivalent to a polynomial in $\mathbf{K}[x_1, x_2]$.*

For arbitrary power series, Corollary 1.3 is the best result that one can get.

EXAMPLE 1.4 ([8, p. 220]). Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Then the power series f in $\mathbf{K}\{x_1, x_2, x_3\}$ defined by

$$f = x_3(x_1^2 + x_2^2 + 2x_3)(x_1^2 + x_2^2 + x_3)(x_1^2 + x_2^2 - (1 + x_1)x_3)(x_1^2 + x_2^2 - e^{x_1}x_3)$$

is not equivalent to a polynomial in $\mathbf{K}[x_1, x_2, x_3]$. Note that if $\mathbf{K} = \mathbf{R}$, then 0 is an isolated critical point of f .

If $J(f)$ has height n , i.e. $J(f)$ contains a certain power of the maximal ideal of A_n or, equivalently, $A_n/J(f)$ is a finitely generated vector space over $\mathbf{K}$, then one can obtain a stronger result than Theorem 1.1. We say that a power series f in $m(A_n)$ is weakly finitely determined provided that there exists a positive integer l such that if f is written as $f = \varepsilon f_1^{\lambda_1} \cdots f_p^{\lambda_p}$, where ε belongs to $\mathbf{K}^*$, $f_1, \dots, f_p$ are irreducible, relatively prime elements of A_n and $\lambda_1, \dots, \lambda_p$ are positive integers, then every power series g of the form $g = \varepsilon g_1^{\lambda_1} \cdots g_p^{\lambda_p}$ with $g_j - f_j$ belonging to $m(A_n)^{l+1}$ for $j = 1, \dots, p$ is equivalent to f . In particular, every weakly finitely determined power series is equivalent to a polynomial over $\mathbf{K}$.

THEOREM 1.5. *Let f be in $m(A_n)$. If $A_n/J(f)$ is a finitely generated vector space over $\mathbf{K}$, then f is weakly finitely determined.*

We should mention a geometric interpretation of the ideal $J(f)$.

Let $\mathbf{K}$ be one of the fields $\mathbf{R}$ or $\mathbf{C}$. We shall identify elements of O_n with germs of analytic functions at the origin in $\mathbf{K}^n$. Clearly, two elements f and g in O_n are equivalent if and only if there exists an analytic local diffeomorphism $\sigma: (\mathbf{K}^n, 0) \rightarrow (\mathbf{K}^n, 0)$ such that $g = f \circ \sigma$. Given an ideal I or an element f of O_n , $V(I)$ and $V(f)$

will denote the set-germs at the origin of zeros of I and f , respectively. Let V be an analytic set-germ at the origin in $\mathbf{K}^n$ of pure codimension one and let $V_1, \dots, V_p$ be its irreducible analytic components. Let W_j be a representative of V_j . Denote by $S(V)$ the germ at the origin of the set of points x in $W = W_1 \cup \dots \cup W_p$ such that W is not a normal crossing at x or x is a singular point of W_j for some $j = 1, \dots, p$. Let h_j be a generator of the ideal of O_n of power series vanishing on V_j . Note that the ideal $J(V) = I(h_1, \dots, h_p)$ does not depend on the choice of generators. Now let $f: (\mathbf{K}^n, 0) \rightarrow (\mathbf{K}, 0)$ be a germ of an analytic function. Write $f = \varepsilon f_1^{\lambda_1} \cdots f_p^{\lambda_p}$, where ε belongs to $\mathbf{K}^*$, $f_1, \dots, f_p$ are irreducible, relatively prime analytic germs and $\lambda_1, \dots, \lambda_p$ are positive integers. Pick a neighborhood U of the origin in $\mathbf{K}^n$ and a representative $g_j: U \rightarrow \mathbf{K}$ of f_j for $j = 1, \dots, p$. Denote by $S(f)$ the germ at the origin of the set of points x in U such that the function $g = \varepsilon g_1^{\lambda_1} \cdots g_p^{\lambda_p}$ is not a normal crossing at x or $g_j(x) = 0$ and g_j has a critical point at x for some $j = 1, \dots, p$. We recall that an analytic subset W of $\mathbf{K}^n$ is said to be a normal crossing at a point x in W if its germ at x is analytically isomorphic to the germ at the origin of the set $\{(y_1, \dots, y_n) \in \mathbf{K}^n \mid y_1 \cdots y_i = 0\}$ for some $i = 1, \dots, n$. Similarly, an analytic function $g: U \rightarrow \mathbf{K}$, defined on an open set U of $\mathbf{K}^n$, is called a normal crossing at a point x in U if there exists a local analytic coordinate system $(U_x, y_1, \dots, y_n)$ with origin x , a nonvanishing analytic function $u: U_x \rightarrow \mathbf{K}$ and nonnegative integers $k_1, \dots, k_n$ such that $g = uy_1^{k_1} \cdots y_n^{k_n}$ on U_x .

PROPOSITION 1.6. *Let $\mathbf{K}$ be one of the fields $\mathbf{R}$ or $\mathbf{C}$. For any f in $m(O_n)$ one has $V(J(f)) = S(f)$. Moreover, if $\mathbf{K} = \mathbf{C}$, then $J(f) = J(V(f))$, $V(J(f)) = S(V(f))$ and the height of $J(f)$ is equal to the codimension of $S(V(f))$.*

PROOF. An exercise. The last part follows from the Nullstellensatz.

As we saw, in the complex case, the ideal $J(f)$ is completely determined by $V(f)$. From Theorem 1.1, one obtains immediately:

COROLLARY 1.7. *Let f be in $\mathbf{C}\{x_1, \dots, x_n\}$ and $f(0) = 0$. If $\text{codim } S(V(f)) = n - k$, then f is equivalent to a polynomial in $x_{k+1}, \dots, x_n$ with coefficients in $\mathbf{C}\{x_1, \dots, x_k\}$.*

Over the field $\mathbf{R}$ information about $V(f)$ is, usually, insufficient to draw some conclusions about f . Indeed, f can have factors vanishing only at the origin, $V(f)$ need not be coherent, etc. Clearly, $S(V(f)) \subseteq S(f)$ but, in general, these two germs are different. Moreover, the height of $J(f)$ is less than or equal to the codimension of $V(J(f)) = S(f)$ but the strong inequality can occur. For instance, taking f of Example 1.4, one has $S(f) = \{0\}$ but f is not equivalent to a polynomial over $\mathbf{R}$ and thus, according to Theorem 1.1, the height of $J(f)$ is less than 3. Nevertheless, we have the following.

THEOREM 1.8. *Let f be in $\mathbf{R}\{x_1, \dots, x_n\}$, $f(0) = 0$, and let r be a nonnegative integer. If the codimension of $S(f)$ is equal to $n - k$, then there exists a local $\mathbf{C}^r$ diffeomorphism $\sigma: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ such that $f \circ \sigma$ is a polynomial in $x_{k+1}, \dots, x_n$ with coefficients in $\mathbf{R}\{x_1, \dots, x_k\}$.*

It follows from Example 1.4 that σ cannot be replaced by an analytic diffeomorphism.

Our method also works for C^∞ functions. Let E_n be the ring of germs at the origin of C^∞ functions on $\mathbf{R}^n$ and let T be the homomorphism from E_n onto $\mathbf{R}[[x_1, \dots, x_n]]$ induced by the Taylor expansion at the origin. We say that an element f in E_n is factorizable if it can be written as $f = \varepsilon f_1^{\lambda_1} \cdots f_p^{\lambda_p}$, where ε is in $\mathbf{R}^*$, $\lambda_1, \dots, \lambda_p$ are positive integers, $f_1, \dots, f_p$ belong to E_n and $Tf_1, \dots, Tf_p$ are irreducible, relatively prime elements of $\mathbf{R}[[x_1, \dots, x_n]]$. Two function-germs f and g in E_n are said to be equivalent if there exists a local C^∞ diffeomorphism $\sigma: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ such that $g = f \circ \sigma$.

THEOREM 1.9. *Let f be a factorizable element of E_n . If the height of the ideal $J(Tf)$ is equal to $n - k$, then f is equivalent to a polynomial in $x_{k+1}, \dots, x_n$ with coefficients in E_k .*

From Theorem 1.9 and Lemma 1.2 one obtains immediately

COROLLARY 1.10. *Every factorizable element in E_n is equivalent to a polynomial in x_{n-1}, x_n with coefficients in E_{n-2} . In particular, an element f of E_2 is equivalent to a polynomial in $\mathbf{R}[x_1, x_2]$ if and only if $f - f(0)$ is factorizable.*

To trace the history of the problem the reader may consult [4, 5, 8, 10]. The main results, Theorems 1.1, 1.8, 1.9 and Corollary 1.7, are new and considerably stronger than the ones previously known. Corollary 1.3, for $\mathbf{K} = \mathbf{R}$ and for convergent power series over $\mathbf{K} = \mathbf{C}$ as well as Corollary 1.10 have been proved in [7]. The ideal $J(f)$ has been introduced in the paper [2], which also contains a proof of Theorem 1.5 for convergent power series over $\mathbf{R}$ or $\mathbf{C}$. The second part of Corollary 1.3 and a different version of Theorem 1.5, both for convergent power series over $\mathbf{C}$, are in [3]. It should be mentioned that the proofs in [2, 3, and 7] do not carry over on power series over an arbitrary field of characteristic 0 (not even on formal power series over $\mathbf{C}$). Moreover, what perhaps is more interesting, the proofs presented in this note are simpler even for $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$.

2. Some ideals associated with power series. Let $\varphi = (\varphi_1, \dots, \varphi_p)$ be a p -tuple of elements of A_n and let $\lambda = (\lambda_1, \dots, \lambda_p)$ be a p -tuple of positive integers. We denote by $I(\varphi; \lambda)$ the ideal of A_n generated by all $(p+1) \times (p+1)$ minors of the $(p+1) \times (p+n)$ matrix

$$M(\varphi; \lambda) = \begin{pmatrix} \varphi_1 & 0 & \cdots & 0 & \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ 0 & \varphi_2 & \cdots & 0 & \frac{\partial \varphi_2}{\partial x_1} & \cdots & \frac{\partial \varphi_2}{\partial x_n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \varphi_p & \frac{\partial \varphi_p}{\partial x_1} & \cdots & \frac{\partial \varphi_p}{\partial x_n} \\ \lambda_1 & \lambda_2 & \cdots & \lambda_p & 0 & \cdots & 0 \end{pmatrix}.$$

The following observation will play an important role.

LEMMA 2.1. *Let $\varphi = (\varphi_1, \dots, \varphi_p)$ be a p -tuple of elements of $m(A_n)$ and let $\lambda = (\lambda_1, \dots, \lambda_p)$ be a p -tuple of positive integers. Then $I(\varphi; \lambda)$ is contained in $I(\varphi)$ and the radicals of both ideals are equal.*

PROOF. Since the ring F_n is flat over O_n , it suffices to prove the lemma for $A_n = F_n$. The first part of the conclusion is obvious. To show the second part we may assume that $\mathbf{K}$ is an algebraically closed field. Indeed, the general case follows easily by passing to the ring of formal power series over the algebraic closure of $\mathbf{K}$. By [10, p. 49], we only have to prove that if $H: F_n \rightarrow \mathbf{K}[[t]]$ is a $\mathbf{K}$ -homomorphism of algebras vanishing on $I(\varphi; \lambda)$, then it also vanishes on $I(\varphi)$. Let $f = \varphi_1^{\lambda_1} \cdots \varphi_p^{\lambda_p}$. Note that for each $i = 1, \dots, n$,

$$\frac{\partial f}{\partial x_i} = \varphi_1^{\lambda_1-1} \cdots \varphi_p^{\lambda_p-1} \left(\lambda_1 \frac{\partial \varphi_1}{\partial x_i} \varphi_2 \cdots \varphi_p + \cdots + \lambda_p \varphi_1 \cdots \varphi_{p-1} \frac{\partial \varphi_p}{\partial x_i} \right)$$

belongs to $I(\varphi; \lambda)$ and hence, by [10, p. 50], $H(f) = 0$. Clearly, the rows of the matrix $M(\varphi; \lambda)_H$ obtained by applying H to each entry in $M(\varphi; \lambda)$ are linearly dependent over the field $\mathbf{L}$ of fractions of $\mathbf{K}[[t]]$. Since $H(\varphi_j) = 0$ for some $j = 1, \dots, p$, the last row $(\lambda_1, \lambda_2, \dots, \lambda_p, 0, \dots, 0)$ is not a linear combination of the first p rows of $M(\varphi; \lambda)_H$. Thus the first p rows of $M(\varphi; \lambda)_H$, which are the same as the rows of the matrix obtained by applying H to each entry in $M(\varphi)$, are linearly dependent over $\mathbf{L}$ and H vanishes on $I(\varphi)$.

REMARK 2.2. For convergent power series over $\mathbf{C}$ one can also prove Lemma 2.1 by applying the classical Nullstellensatz.

3. The proofs. We only have to prove Theorems 1.1, 1.5, 1.8, and 1.9.

PROOF OF THEOREM 1.1. Let $f = \varepsilon f_1^{\lambda_1} \cdots f_p^{\lambda_p}$, where ε is in $\mathbf{K}^*$, $f_1, \dots, f_p$ are irreducible, relatively prime elements in A_n and $\lambda_1, \dots, \lambda_p$ are positive integers. Let $\varphi = (f_1, \dots, f_p)$ and $\lambda = (\lambda_1, \dots, \lambda_p)$. By Lemma 2.1, the height of the ideal $I = m(A_n)^2 I(\varphi; \lambda)^2$ is equal to $n - k$. Now, linearly changing variables and applying the normalization theorem for power series, we may assume that the ideal generated by $x_1, \dots, x_k$ and I contains a certain power of $m(A_n)$. Hence $I + O_k[x_{k+1}, \dots, x_n] = O_n$. In particular, one can find elements $g_1, \dots, g_p$ in $O_k[x_{k+1}, \dots, x_n]$ such that $f_j - g_j$ belongs to I for $j = 1, \dots, p$. Now set $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_p)$, $z = (z_1, \dots, z_n)$ and consider $F = (F_1, \dots, F_p, F_{p+1})$ defined by

$$F_j(x, y, z) = (1 + y_j)f_j(x + z) - g_j(x)$$

for $j = 1, \dots, p$ and

$$F_{p+1}(x, y, z) = (1 + y_1)^{\lambda_1} \cdots (1 + y_p)^{\lambda_p} - 1.$$

The Jacobian matrix $F'_{(y,z)}(x, 0, 0)$ is equal to $M(\varphi; \lambda)$. By Tougeron's implicit function theorem [10, p. 57], there exist power series $y(x) = (y_1(x), \dots, y_p(x))$ and $z(x) = (z_1(x), \dots, z_n(x))$ such that $F(x, y(x), z(x)) = 0$ and $y_j(x), z_i(x)$ belong to $m(A_n)^2$ for $j = 1, \dots, p$, $i = 1, \dots, n$. Now $\sigma(f) = g$, where $g = \varepsilon g_1^{\lambda_1} \cdots g_p^{\lambda_p}$ and σ is the $\mathbf{K}$ -automorphism of A_n defined by $\sigma(h(x)) = h(x + z(x))$ for $h(x)$ in A_n .

PROOF OF THEOREM 1.5. Let $f = \varepsilon f_1^{\lambda_1} \cdots f_p^{\lambda_p}$, where ε belongs to $\mathbf{K}^*$, $f_1, \dots, f_p$ are irreducible, relatively prime elements of A_n and $\lambda_1, \dots, \lambda_p$ are positive integers. Thanks to Lemma 2.1, one can find a positive integer l such that $m(A_n)^{l+1}$ is contained in $m(A_n)^2 I(\varphi; \lambda)^2$, with $\varphi = (f_1, \dots, f_p)$ and $\lambda = (\lambda_1, \dots, \lambda_p)$. Let $g_1, \dots, g_p$ be elements of A_n such that $g_j - f_j$ belongs to $m(A_n)^{l+1}$ for $j = 1, \dots, p$. Now, by considering $F = (F_1, \dots, F_p, F_{p+1})$ defined as in the proof of Theorem 1.1, the conclusion follows.

PROOF OF THEOREM 1.9. Let $f = \varepsilon f_1^{\lambda_1} \cdots f_p^{\lambda_p}$, where ε is in $\mathbf{R}^*$, $\lambda_1, \dots, \lambda_p$ are positive integers, $f_1, \dots, f_p$ belong to E_n and $Tf_1, \dots, Tf_p$ are irreducible relatively prime elements of $\mathbf{R}[[x_1, \dots, x_n]]$. Let $\varphi = (f_1, \dots, f_p)$, $\lambda = (\lambda_1, \dots, \lambda_p)$ and let $I(\varphi; \lambda)$ be the ideal of E_n defined in the same way as the corresponding ideal in the case of power series. Denote by $m(E_n)$ the maximal ideal of E_n . By Lemma 2.1, the ideal $T(I)$ of $\mathbf{R}[[x_1, \dots, x_n]]$, where $I = m(E_n)^2 I(\varphi; \lambda)^2$, has height $n - k$. Thanks to [10, p. 168], we may assume that the coordinates have been chosen such that there exist $g_1, \dots, g_p$ which are polynomials in $x_{k+1}, \dots, x_n$ with coefficients in E_k and $f_j - g_j$ belongs to I for $j = 1, \dots, p$. Let $F = (F_1, \dots, F_p, F_{p+1})$ be defined as in the proof of Theorem 1.1. The conclusion follows by applying Tougeron's implicit function theorem for C^∞ functions [8].

In the proof of Theorem 1.8 it will be convenient to identify set-germs and function-germs with their representatives.

PROOF OF THEOREM 1.8. Write $f = \varepsilon f_1^{\lambda_1} \cdots f_p^{\lambda_p}$, where ε is in $\mathbf{R}^*$, $f_1, \dots, f_p$ are irreducible, relatively prime elements of O_n and $\lambda_1, \dots, \lambda_p$ are positive integers. Let $\varphi = (f_1, \dots, f_p)$ and $\lambda = (\lambda_1, \dots, \lambda_p)$. Choose an element h in the ideal $I = m(O_n)^2 I(\varphi; \lambda)^2$ such that $h \geq 0$ and $V(h) = V(I)$. By the Lojasiewicz inequality [10], there exist positive real numbers c, η and α such that

$$h(x) \geq cd(x, V(h))^\alpha \quad \text{for } \|x\| < \eta,$$

where $d(x, V(h))$ denotes the distance from x to $V(h)$. Now take a positive integer l such that for each analytic function u defined in a neighborhood of 0 in $\mathbf{R}^n$ and l -flat on $V(h)$, the function u/h extends to a C^{r+3} function on a neighborhood of 0 vanishing on $V(h)$. Let J be the ideal of elements of O_n vanishing on $S(f)$. Clearly, J as well as J^{l+1} have height $n - k$. By Proposition 1.6 and Lemma 2.1, $V(h) = S(f)$. Thanks to the normalization theorem, we may assume that $J^{l+1} + O_k[x_{k+1}, \dots, x_n] = O_n$. Denote by $E_n^{(r+3)}$ the ring of germs at the origin of C^{r+3} functions on $\mathbf{R}^n$. Note that J^{l+1} is contained in $IE_n^{(r+3)}$. Thus one can find elements $g_1, \dots, g_p$ in $O_k[x_{k+1}, \dots, x_n]$ such that $f_j - g_j$ belongs to $IE_n^{(r+3)}$ for $j = 1, \dots, p$. Now consider $F = (F_1, \dots, F_p, F_{p+1})$ defined as in the proof of Theorem 1.1. By applying Tougeron's implicit function theorem for finitely differentiable functions [1, 9], one concludes the proof.

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