JACOBSON RADICAL OF FILTERED ALGEBRAS

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ABSTRACT. Using elementary graded ring theory methods we show that the Jacobson radical of certain filtered algebras is zero. We then use this to propose a simpler and more general proof of one of the main results contained in [1].

Throughout this note, K will be a commutative ring with 1. An *algebra* A will mean a K -module equipped with an associative multiplication, whose identity element is $1 \in K$.

A is a *filtered algebra* if there is an ascending chain of K -submodules of A , namely $(0) = A_{(-1)} \subseteq A_{(0)} \subseteq A_{(1)} \subseteq \cdots$, such that $A_{(i)}A_{(j)} \subseteq A_{(i+j)}$ for all nonnegative integers i and j . We will consider only nontrivial filtrations, i.e. the case when $A \neq A_{(0)}$, and we will always assume that $1 \in A_{(0)}$. For each element a of A we define $d(a)$ to be the smallest integer i for which $a \in A_{(i)}$.

A filtration of a K -algebra A gives rise to a corresponding graded object, the *associated graded algebra* of A , defined as the K -direct sum $\text{Gr}(A) = A_{(0)}/A_{(-1)} \oplus A_{(1)}/A_{(0)} \oplus A_{(2)}/A_{(1)} \oplus \cdots$ in which multiplication is given by

$$(a + A_{(i-1)})(b + A_{(j-1)}) := ab + A_{(i+j-1)} \quad \text{for any } i, j \geq 0,$$

with the above rule extended by linearity. For an element $a \neq 0$ of the filtered algebra A we define $\text{gr}(a)$ to be the element $a + A_{(d(a)-1)}$ of $\text{Gr}(A)$. We also declare that $\text{gr}(a) = 0$ if and only if $a = 0$.

$J(A)$ will denote the Jacobson radical of A .

THEOREM. *Let A be an algebra filtered in such a way that $\text{Gr}(A)$ has no zero divisors. Then $J(A) = (0)$.*

PROOF. Take any $a \in J(A)$. There exists a nonzero $b \in A$ for which $(1 - a)b = 1$. Suppose that $d(a) > 0$. Then $\text{gr}(1 - a) = \text{gr}(a)$, and it follows that $\text{gr}(a)\text{gr}(b) = \text{gr}(1 - a)\text{gr}(b) = 1 + A_{(d(a)+d(b)-1)} = 0$. Since both $\text{gr}(a)$ and $\text{gr}(b)$ are nonzero, we have a contradiction with the hypothesis. Hence $d(a) \leq 0$, meaning that $J(A) \subseteq A_{(0)}$. Now let $b \in A$ with $d(b) > 0$, and let a be an element of $J(A)$ once again. Then $ab \in J(A) \subseteq A_{(0)}$, so that $\text{gr}(a)\text{gr}(b) = ab + A_{(d(a)+d(b)-1)}$ is zero (since $d(a) + d(b) - 1 \geq 0$). This implies that $a = 0$, thus proving the assertion. $\square$

The above theorem allows us to generalize Theorem 1.1 of [1].

COROLLARY. *Let B be the universal enveloping algebra of a Lie algebra L . Then $J(B) = N \cdot B$, where N is the nil radical of K .*

Received by the editors November 4, 1985.

1980 *Mathematics Subject Classification.* Primary 16A20; Secondary 17B35.

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 0002-9939/86 \$1.00 + \$.25 per page

PROOF. We follow the original argument given in [1]. It is well known that in our case N is the intersection of all prime ideals of K . If P is a prime ideal of K , then B/PB is an enveloping algebra over the domain K/P , and by the Poincaré-Birkhoff-Witt theorem it can be filtered in such a way that $\text{Gr}(B/PB)$ is a polynomial ring over K/P . The theorem now implies that $J(B/PB)$ is zero, and hence $J(B) \subseteq \bigcap P \cdot B$, where P runs over all prime ideals of K ; the latter set, by freeness of B over K , is equal to $N \cdot B$. This, combined with the obvious inclusion $N \cdot B \subseteq J(B)$, gives the thesis. $\square$

To conclude, we will mention that the technique used above allows the rank of L over K to be infinite, thus removing one restriction imposed on L in [1]. In addition, the characteristic of K has no relevance in our reasoning at all, making the proof self-contained and simpler than in [1]. However, for this we pay the price of not being able to say anything about the primitive spectrum of $U(L)$, such information being an important by-product of the method employed in [1].

We would like to thank L. W. Small for his advice and stimulating comments, and S. Montgomery for pointing out that similar arguments are used in [2, 3] where she considers filtered rings whose associated graded rings have no zero divisors. Last but not least, we are grateful to R. S. Irving for his valuable and encouraging remarks.

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