POLYNOMIAL RINGS OVER GOLDIE RINGS ARE OFTEN GOLDIE

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ABSTRACT. Here, we prove a result that has as a consequence the fact that if the ring R is an algebra over an uncountable field the a.c.c. on annihilators is preserved under polynomial extensions by any number of variables.

Recently Jeanne Kerr [1] has given an example of a ring R with ascending chain condition on annihilators that has the property that the polynomial ring $R[X]$ does not have the ascending chain condition on annihilators (a.c.c.). This answered a question of long standing duration on the behavior of the classical Goldie conditions under polynomial extensions. Kerr's example is an algebra over Z_2 .

LEMMA. *Let R be a ring which contains an uncountable set V in the center of R having the property that if $u, v \in V$ and $u \neq v$, then $u - v$ is a nonzero divisor. Let S be a countable subring of R . Then, there is an infinite subset V' of V such that V' is algebraically independent over S .*

PROOF. Let V' be a maximal subset of V which is algebraically independent over S . Suppose V' is finite. Then, any $v \in V$ satisfies a nonzero polynomial in the ring $(S[V'])[X]$. Since there are uncountably many elements in V and only countably many polynomials, there is a polynomial $f(x)$ with infinitely many roots in V . Let u and v be two of these. Since we may divide by monics in any ring, $f(x) = (x - u)h(x)$, but $f(v) = 0 = (v - u)h(v)$ and $v - u$ is regular so $h(v) = 0$, and continuing, we obtain a contradiction since the number of roots is larger than the degree of $f(x)$.

THEOREM. *Let R be a ring containing an uncountable set V in the center of R , so that if u and v are distinct elements of V , $u - v$ is a nonzero divisor. Let P be a property so that*

- (*) *A ring T satisfies P if and only if every countable subring of T satisfies P (e.g., the a.c.c. on annihilators).*

Then R satisfies P if and only if $R[X]$ satisfies P , where X is any set of variables.

PROOF. One way is trivial. Conversely, we show $R[X]$ satisfies P by showing that every countable subring R_0 of $R[X]$ does. Clearly, $R_0 \subset S[x_1, x_2, \dots]$ for some countable subring S of R and countable set of variables. Choose V' as in the lemma so that $S[x_1, x_2, \dots] \approx S[V'] \subset R$. Then, since $R_0 \subset S[x_1, x_2, \dots]$, R_0 has P , so that by (*) $R[X]$ does.

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COROLLARY. *If R is an algebra over an uncountable field and X is any set of variables then R has the ascending chain condition on annihilators if and only if $R[X]$ does.*

REMARK. The natural proof of the above, say in one variable, consists in the following: Any statement about chains of annihilators involves only countably many polynomials. Now, find an element in the field on which none of these vanish and embed the problem in R by evaluation at this element.

Schock [2] proved that finite dimensionality is preserved by polynomial extensions, so that

COROLLARY. *If R is an algebra over an uncountable field, R is Goldie if and only if $R[X]$ is Goldie.*

REFERENCES

1. Jeanne Kerr, (to appear).
2. Robert Schock, *Polynomial rings over finite dimensional rings*, Pacific J. Math. **42** (1972), 251–257.

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