ON A PROBLEM OF MAGNUS

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In honor of Boris Weisfeiler

ABSTRACT. By using “strong approximation” results for linear group a question of Magnus on residual properties of free groups is answered affirmatively.

Let $\mathcal{C}$ be a nonempty class of groups and Γ a group. Γ is said to be residually- $\mathcal{C}$ if for every $g \neq 1$ in Γ there is an epimorphism $\phi: G \rightarrow C$ where C is a group belonging to $\mathcal{C}$ s.t. $\phi(g) \neq 1$. Equivalently the intersection of all normal subgroups N of Γ for which Γ/N belongs to $\mathcal{C}$ is the unit element of G .

The residual properties of the free groups F_n received a lot of attention: They had been known to be residually finite even before P. Hall coined the phrase “residually finite”. They are also known to be residually- $\mathcal{C}(p)$ where $\mathcal{C}(p)$ is the class of finite p -groups for a fixed prime p .

In 1968 Wilhelm Magnus surveyed the state of the art of residually finite groups and wrote [M, p. 309]:

“For $n \geq 2$, F_n is not only residually- $\mathcal{C}(p)$ for every prime number p , it is also residually- A and residually- $\text{PSL}(2, p^k)$ for fixed k and variable p , where A denotes the class of alternating groups Here, incidentally the story ends. It is not even known whether F_2 is residually- $\text{PSL}(3, p)$ where p runs through all primes or residually- $\text{PSL}(2, 2^k)$ where k runs through the positive integers.”

In this note we observe that these two questions can be answered affirmatively by using recent deep “strong approximation” theorems for linear groups proved by Weisfeiler [W] (see also Nori [N] and Matthews-Vaserstein-Weisfeiler [MVW]).

These results say, for example, that if Γ is a subgroup of $\text{SL}(3, \mathbf{Z})$ which is Zariski dense in $\text{SL}(3, \mathbf{C})$, then its closure in the congruence topology of $\text{SL}(3, \mathbf{Z})$ is of finite index in $\text{SL}(3, \mathbf{Z})$.

In particular, for almost all primes p , Γ is mapped onto $\text{SL}(3, p)$ under the canonical map $\phi_p: \text{SL}(3, \mathbf{Z}) \rightarrow \text{SL}(3, \mathbf{Z}/p\mathbf{Z})$. The intersection of all kernels of ϕ_p , for an infinite set of primes, is trivial and so Γ is residually- $\{\text{SL}(3, p) \mid p \text{ prime}\}$ (and also residually- $\text{PSL}(3, p)$ if the center of Γ is trivial). To answer Magnus’ first question it remains now to check that F_2 can be embedded as a Zariski-dense subgroup in $\text{SL}(3, \mathbf{Z})$. Indeed, $\text{SL}(3, \mathbf{Z})$ is Zariski-dense in $\text{SL}(3, \mathbf{C})$ by Borel Density

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Theorem [R, Chapter V] and any Zariski dense subgroup contains a Zariski-dense subgroup isomorphic to F_2 by a famous result of Tits [T, Theorem 3].

For the second case $\mathcal{G} = \{\mathrm{PSL}(2, 2^k) \mid k \in \mathbf{N}\}$ again it suffices to prove for $\mathcal{G} = \{\mathrm{SL}(2, 2^k) \mid k \in \mathbf{N}\}$ and we argue essentially the same way: This time use Theorem 4 of [T] to embed F_2 as a Zariski-dense subgroup of $\mathrm{SL}_2(\mathbf{F}_2[X])$ (where $\mathbf{F}_2[X]$ is the polynomial ring over the field of order 2, $\mathbf{F}_2$) and then use Theorem 8.2 of [W] to conclude that $\Gamma = F_2$ is mapped onto $\mathrm{SL}_2(\mathbf{F}_2[X]/\mathbf{M})$ for almost every maximal ideal $\mathbf{M}$ of $\mathbf{F}_2[X]$. But $\mathbf{F}_2[X]/\mathbf{M} \simeq \mathbf{F}_{2^k}$ for some k . So F_2 is residually- $\mathrm{SL}_2(2^k)$. This settles the second question of Magnus.

The same arguments work in a much more general content.

THEOREM. *Let G be a Chevalley group scheme defined over $\mathbf{Z}$ (e.g., $G = \mathrm{SL}(n)$, $\mathrm{Sp}(2n)$ etc.). Then*

- (a) *For a fixed k and variable p , F_2 is residually- $\{G(p^k) \mid p \text{ prime}\}$.*
- (b) *For a fixed p and variable k , F_2 is residually- $\{G(p^k) \mid k \in \mathbf{N}\}$ unless $p = 2$ and G is of type B_n , C_n or F_4 or $p = 3$ and G is of type G_2 .*

Here $G(l)$ denotes the K -rational points of G over the field K of order l .

For the proof of (a): Let L be a Galois extension of $\mathbf{Q}$ of degree k such that there exists infinitely many primes p in $\mathbf{Z}$ which are still primes in L , i.e. the ideal $p\mathcal{O}$ of the ring of algebraic integers $\mathcal{O}$ of L is a maximal ideal of $\mathcal{O}$. Such an L does exist: For example, let q be a prime such that $q \equiv 1 \pmod{k}$ and let M be the cyclotomic field $\mathbf{Q}[\exp(2\pi i/q)]$. Then $\mathrm{Gal}(M/\mathbf{Q})$ is cyclic of order $q-1$, and we can find therefore a Galois extension $L/\mathbf{Q}$ of degree k such that $L \subseteq M$. It suffices to check that infinitely rational primes are still primes in M . By [C, Theorem 10.45] this is indeed the case for every prime p which is primitive mod $q-1$. There are infinitely many such primes by the Dirichlet theorem.

So $\mathcal{O}$ has infinitely many maximal ideals $\mathbf{M}$ for which $\mathcal{O}/\mathbf{M}$ is a field of order p^k for some prime p . Now, consider G as defined over L and let $H = \mathrm{Res}_{L/\mathbf{Q}}(G)$ be the $\mathbf{Q}$ -algebraic group obtained by “restriction of the scalars” (cf. [Z, p. 116]). So $H(\mathbf{Z}) = G(\mathcal{O})$ (loc. cit.). Again, use the Borel density theorem and [T, Theorem 3] to find a Zariski dense subgroup Γ of $H(\mathbf{Z})$ isomorphic to F_2 . By [W, Theorem 9.1] (or Nori [N] or [MWV, p. 515]) Γ is mapped onto $H(\mathbf{Z}/p\mathbf{Z})$ for almost all p . But $G(\mathcal{O}/\mathbf{M}) = H(\mathbf{Z}/p\mathbf{Z})$ where $\mathbf{M}$ is the ideal $p\mathcal{O}$, so Γ is mapped onto $G(p^k)$ for infinitely many primes p . The kernel is clearly trivial so Γ is residually- $\{G(p^k) \mid p \text{ prime}\}$.

The proof of the second part of the theorem is just the same as the proof for $\mathrm{SL}(2, 2^k)$ except that this time we take $G(\mathbf{F}_p[X])$. We have to exclude few cases in order to be able to apply Theorem 9.1 of [W]. It is very likely that the theorem holds for these cases as well.

REMARKS. (1) It is interesting to note that the proofs in [MVW and W] use the classification of the finite simple groups (while Nori’s proof does not, but Nori’s result covers only our needs for part (a) of the theorem and not for part (b)). So our theorem can also be considered as a by-product of the classification.

(2) Every free group is residually- $\{F_2\}$ (cf. [M]), so our theorem holds for every free group. (This can also be seen from the proof.)

(3) The theorem says in particular that for fixed n and k and variable p , and for fixed n and p and variable k , F_2 is residually $\text{PSL}(n, p^k)$. But, it leaves open the question whether F_2 is residually $\text{PSL}(n, p^r)$ for p and r fixed and n variable. One can also speculate and ask whether F_2 is residually every infinite set of finite simple groups.

ADDED IN PROOF. 1. Since this paper was written, I have learned from O. Kegel, M. Newman and S. Pride about other special cases of the Theorem in the literature. See [1, 2] and the references therein.

2. J. S. Wilson has answered the first question in Remark (3) affirmatively. The second was previously raised by Yu. M. Gorchakov (cf. [2]).

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2. V. M. Levchuk and Yu. W. Nuzlin, *Structure of Ree groups*, Algebra and Logic **24** (1985), 16–23.

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