ON THE STABLE RANGE OF UNIFORM ALGEBRAS AND H^∞

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ABSTRACT. We find a criterion for stable range reducibility for uniform algebras and we apply it to the algebra H^∞ .

The notion of stable rank of a ring introduced by Bass [1], when applied to a commutative Banach algebra A , is strongly related with some homotopical invariants of its spectrum $X(A)$ and of the zero-sets of its ideals [2, 3]. An n -tuple $a \in A^n$ is *unimodular* if $A = \sum_{i=1}^n Aa_i$; a is *reducible* if $(a_1 + x_1a_n, \dots, a_{n-1} + x_{n-1}a_n)$ is unimodular for some $x_1, \dots, x_{n-1} \in A$. The *stable rank* of A is the least n such that every unimodular $a \in A^{n+1}$ is reducible.

In [3] we conjectured that the algebra H^∞ of all bounded holomorphic functions on the open unit disc D has stable rank one, i.e. that for every $f, g \in H^\infty$ such that $|f| + |g| \geq \delta_1$ for some $\delta_1 > 0$, there exists $h \in H^\infty$ with $|f + hg| \geq \delta_2$ for some $\delta_2 > 0$. The problem is still open. In this note, however, we prove that every unimodular (f, g) , with f in the disc algebra, is reducible. The proof of this result (Theorem 2) is based on some work of D. J. Newman, used by L. Carleson for proving his corona theorem, and on a result (Theorem 1) which asserts that if A is a uniform algebra and $\alpha, \beta \in A$ are such that $(\alpha - z_i, \beta)$ is reducible for suitable complex numbers z_i , then every unimodular of the form $(F(\alpha), \beta)$, where $F \in R(\text{sp } \alpha)$, is reducible (recall that if K is a compact subset of the plane, $R(K)$ is the algebra of uniform limits of rational functions with poles outside K).

NOTATIONS. Given a complex commutative Banach algebra, the spectrum $X(A)$ is the space of all nonzero homomorphisms $A \rightarrow \mathbb{C}$ with the topology of simple convergence; it is a compact Hausdorff space and the Gelfand transform $\hat{a}$ of an element a of A is the continuous function $\hat{a}: X(A) \rightarrow \mathbb{C}$ defined by $\hat{a}(h) = h(a)$ ($h \in X(A)$). The image of $\hat{a}$ coincides with the spectrum of a , $\text{sp } a = \{\lambda \in \mathbb{C}; a - \lambda \text{ is not invertible}\}$. We shall use several times the notation Z_a for the closed set $\{h \in X(A); h(a) = 0\}$. Throughout this paper D denotes the open unit disc $\{z \in \mathbb{C}; |z| < 1\}$. H^∞ is the algebra of all bounded holomorphic functions on D with the sup norm. Its spectrum is not known (see Hoffman [5] for some of the known facts about it); given $w \in T = \{z \in \mathbb{C}; |z| = 1\}$, $M_w = \{h \in X(H^\infty); h(z) = w\}$ (where z is the coordinate function).

Let A be a commutative ring with identity. An element (α, β) of A^2 is *unimodular* if $A\alpha + A\beta = A$; $U_2(A)$ is the set of all unimodular elements of A^2 . We say that $(\alpha, \beta) \in U_2(A)$ is *reducible* (in the sense of Bass) if there exists $\gamma \in A$ such that $\alpha + \gamma\beta$ is invertible. For example, if α is invertible, (α, β) is unimodular and

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reducible for every $\beta \in A$ (take $\gamma = 0$). It is easy to prove that

(1) $(\alpha, \beta) \in U_2(A)$ is reducible if and only if $\alpha \in A' + A\beta$ (A' is the group of invertible elements of A);

(2) if $(\alpha, \beta), (\alpha', \beta) \in U_2(A)$ are reducible, then $(\alpha\alpha', \beta) \in U_2(A)$, and $(\alpha\alpha', \beta)$ is reducible, too.

PROPOSITION. *Let A be a commutative Banach algebra with identity. For every $\beta \in A$ let $H_1(A; \beta) = \{\alpha \in A; (\alpha, \beta) \in U_2(A)\}$. Then $A' + A\beta$ is open and closed in $H_1(A; \alpha)$. In particular, if $f: [0, 1] \rightarrow H_1(A; \beta)$ is a continuous curve and $(f(0), \beta)$ is reducible, then $(f(1), \beta)$ is reducible, too.*

PROOF. It is clear that $A' + A\beta$ and $H_1(A; \beta)$ are open in A^2 . So it suffices to prove that $A' + A\beta$ is closed in $H_1(A; \beta)$. Let $u_n \in A'$, $x_n \in A$ such that $u_n + x_n\beta$ converges to $\alpha \in H_1(A; \beta)$. Then there exist $a, b \in A$ such that $a\alpha + b\beta = 1$. Now, A' is open, so $a(u_n + x_n\beta) + b\beta \in A'$ for n large enough, so $a = u + x\beta$ for some $u \in A'$ and $x \in A$; thus, $1 = a\alpha + b\beta = u\alpha + x\alpha\beta + b\beta = u\alpha + (x\alpha + b)\beta$ and $\alpha = u^{-1} - u^{-1}(x\alpha + b)\beta \in A' + A\beta$.

REMARKS. For each positive integer, $U_n(A)$ can be defined as the set of all $a = (a_1, \dots, a_n) \in A^n$ such that $\sum_{i=1}^n Aa_i = A$. There is also an n -dimensional version of reducibility which leads to the stable range conditions of Bass [1]. For commutative rings the statements (1) and (2) above hold replacing $U_2(A)$ by $U_{n+1}(A)$, A' by $U_n(A)$, and A by A^n . The proposition can be generalized, too (see Corach and Suárez [3, Lemma 3.7]), but we do not know if $U_n(A) + A^n\alpha$ is closed in $H_n(A; \alpha)$ when A is noncommutative. If this happens, several facts on the stable rank of Banach algebras, which hold in the commutative case, could be generalized (see Corach and Larotonda [2] and Corach and Suárez [3]).

THEOREM 1. *Let A be a uniform algebra and $\alpha, \beta \in A$. Let D_i , $i \in I$, be the connected components of $\mathbb{C} \setminus \hat{\alpha}(Z_\beta)$. Suppose that, for each $i \in I$, there exists $z_i \in D_i$ such that $(\alpha - z_i, \beta) \in U_2(A)$ is reducible. Then for every $F \in R(\text{sp } \alpha)$ such that $(F(\alpha), \beta) \in U_2(A)$ it holds that $(F(\alpha), \beta)$ is reducible.*

PROOF. For every $z \in \mathbb{C} \setminus \hat{\alpha}(Z_\beta)$, $(\alpha - z, \beta)$ is reducible. In fact, if $\gamma: [0, 1] \rightarrow D_i \subset \mathbb{C} \setminus \hat{\alpha}(Z_\beta)$ is an arc joining z_i and z , it is clear that $(\alpha - \gamma(t), \beta) \in U_2(A) \forall t \in [0, 1]$, so $\alpha - z_i$ and $\alpha - z$ belong to the same component of $H_1(A; \alpha)$.

Thus, for every $z \in \mathbb{C} \setminus \hat{\alpha}(Z_\beta)$ there exists $\mu \in A'$ such that $\hat{\alpha} - z = \hat{\mu}$ on Z_β .

Let $(F(\alpha), \beta) \in U_2(A)$ for some $F \in R(\text{sp } \alpha)$. Then $Z_F \cap \hat{\alpha}(Z_\beta) = \emptyset$, for $(F(\alpha), \beta) \in U_2(A)$ means that $Z_{F(\alpha)} \cap Z_\beta = \emptyset$. In particular, we can find rational functions f_n with poles outside $\text{sp } \alpha$ and zeros outside $\hat{\alpha}(Z_\beta)$ such that $\{f_n\}$ converges uniformly to F on $\text{sp } \alpha$. Then $f_n(\alpha) \rightarrow F(\alpha)$ and $(f_n(\alpha), \beta)$ is reducible, or, what is the same, $f_n(\alpha) \in A' + A\beta$, because $f_n(\alpha)$ is a product of elements which are reducible with β . But $A' + A\beta$ is closed in $H_1(A; \beta)$ by the proposition, so $F(\alpha) \in A' + A\beta$ and $(F(\alpha), \beta)$ is reducible.

COROLLARY 1. *The same conclusion of Theorem 1 holds if every component of $\mathbb{C} \setminus \hat{\alpha}(Z_\beta)$ contains an element of $\mathbb{C} \setminus \text{sp } \alpha$.*

PROOF. For every component D_i of $\mathbb{C} \setminus \hat{\alpha}(Z_\beta)$, let $z_i \in (\mathbb{C} \setminus \text{sp } \alpha) \cap D_i$. Then $\alpha - z_i \in A'$ and $(\alpha - z_i, \beta) \in U_2(A)$ must be reducible. Thus the hypotheses of Theorem 1 are satisfied.

COROLLARY 2. *Let $(\alpha, \beta) \in U_2(A)$ and suppose that $\mathbf{C} \setminus \hat{\alpha}(Z_\beta)$ is connected. Then (α, β) is reducible.*

PROOF. In view of Theorem 1, taking $F(w) = w \ \forall w$ it suffices to exhibit $z \in \mathbf{C} \setminus \hat{\alpha}(Z_\beta)$ with $(\alpha - z, \beta)$ reducible. For this, take $z = \|\alpha\| + 1$; then $z \in \mathbf{C} \setminus \text{im } \hat{\alpha} \subset \mathbf{C} \setminus \hat{\alpha}(Z_\beta)$ and $\alpha - z \in A$, in particular, $(\alpha - z, \beta)$ is reducible.

In [3] it was conjectured that H^∞ has stable rank one, or what is the same, that every $(f, g) \in U_2(H^\infty)$ is reducible. As an approximation to the proof of this conjecture, we have the following result.

THEOREM 2. *Let $(f, g) \in U_2(H^\infty)$. Suppose that f belongs to the disc algebra. Then (f, g) is reducible.*

The proof is largely based on the corona theorem of L. Carleson, particularly on a result due to D. J. Newman, as it appears in [4, Chapter 12]. More precisely, we need the following two lemmas [4, pp. 203–205].

LEMMA A. *Let $B(z) = \prod_{k=1}^s (z - a_k)/(1 - \bar{a}_k z)$ be a finite Blaschke product with distinct zeros $a_1, \dots, a_s$. For $\delta < 1/2$ let F be analytic in the set $\{z; |B(z)| < \delta\}$ and satisfy $|F(z)| < 1$ there. Then there exists $f \in H^\infty$ with $f(a_k) = F(a_k)$, $k = 1, \dots, s$ and $\|f\| \leq \delta^{-\alpha}$, where α is an absolute constant.*

LEMMA B. *Let f be a function in the disc algebra such that $0 < |f(z)| \leq 1$ on $|z| = 1$ and $E = \{z; |z| = 1, |f(z)| < 1\}$ is nonempty. Then there exists a sequence $\{B_n\}$ of finite Blaschke products with simple zeros such that $|B_n(z)| \rightarrow |f(z)|$ uniformly on each closed subset of $\bar{D} \setminus \bar{E}$ and $B_n(z) \rightarrow f(z)$ uniformly on each closed subset of D .*

PROOF OF THEOREM 2. Using a Möbius transformation we may suppose that $g(0) \neq 0$ and $\|g\| = 1$. It follows from Theorem 1 that it suffices to prove that (z, g) is reducible; in fact, $\hat{z}(Z_g) = \{w \in S^1; \text{there exists } \phi \in M_w \text{ such that } \phi(g) = 0\} \cup \{w \in D; g(w) = 0\}$, so $\mathbf{C} \setminus \hat{z}(Z_g)$ has one or two connected components; in the first case, we use Corollary 2 of Theorem 1; in the second case, $(z - w, g)$ is reducible if $|w| > 1$ because $z - w$ is invertible, so if we prove that (z, g) is reducible we apply Theorem 1, observing that $R(\text{sp } \hat{z})$ is the disc algebra.

Let $0 < \delta_0 < 1$ such that $|z| + |g(z)| > \delta_0$ for every $z \in D$. There exists $E \subset S^1 = \{z; |z| = 1\}$ with measure 2π such that g has nontangential limit at every $z \in E$; the limit function will be denoted by g , too.

Let $V = \{z_0/|z_0|; z_0 \in D, g(z_0) = 0\} \cup \{z \in E; g(z) = 0\}$. Then $E \setminus V$ has measure 2π . If $z_1 \in E \setminus V$ there exists $\delta_1 > 0$ such that $|g(rz_1)| > \delta_1$ for $0 \leq r < 1$. Let $R = \{rz_1; 0 \leq r < 1\}$ and $\delta = \min\{\delta_0, \delta_1\}$. Then $|z| + |g(z)| > \delta$ for every $z \in D$ and $|g(z)| > \delta$ for every $z \in R$.

For $h \in H^\infty$ let $S_\delta(h) = \{z \in D; |h(z)| < \delta/2\}$. Consider $g_\rho(z) = g(\rho z)$ for those $\rho < 1$ such that $g(\rho z) \neq 0$ in S^1 . Then $R \subset D \setminus S_\delta(g_\rho)$ and $|z| + |g_\rho(z)| > \delta$, for $|z| + |g_\rho(z)| > |\rho z| + |g_\rho(z)| > \delta$ if $z \in D$. Let ϕ_ρ be a nonvanishing function in the disc algebra such that $|\phi_\rho(z)| = \min\{3/\delta, 1/|g_\rho(z)|\}$ for $z \in S^1$, and let $\psi_\rho(z) = \phi_\rho(z)g_\rho(z)$. Observe that ϕ_ρ can be constructed by means of a Poisson integral and an exponential.

Then the following facts hold:

- (i) $|\phi_\rho(z)| \geq 1$ and $|\psi_\rho(z)| \leq 1$ for every $z \in \bar{D}$,

(ii) for $z \in S^1$, $|\psi_\rho(z)| = 1 \Leftrightarrow |\phi_\rho(z)| = 1/|g_\rho(z)| \Leftrightarrow |g_\rho(z)| \geq \delta/3$,

(iii) for each ρ there exists a sequence B_n of finite Blaschke products with simple zeros such that $B_n \rightarrow \psi_\rho$ uniformly on the compacts of D , and $|B_n(z)| > |\psi_\rho(z)| - \delta/4$ for n sufficiently large if $|g_\rho(z)| \geq \delta/2$ (Lemma B),

(iv) $|z| + |B_n(z)| > \delta/2$ for $z \in D$ (in fact, $|z| + |B_n(z)| > |z| + |\psi_\rho(z)| - \delta/4 \geq |z| + |g_\rho(z)| - \delta/4 > \delta/2$ for $|g_\rho(z)| \geq \delta/2$, and $|z| > \delta/2$ for $|g_\rho(z)| < \delta/2$),

(v) $|B_n(z)| \geq \delta/4$ for every $z \in R$ (for $R \subset \{z; |g_\rho(z)| \geq \delta/2\}$ and, by (iii), $|B_n(z)| > |\psi_\rho(z)| - \delta/4 \geq |g_\rho(z)| - \delta/4 \geq \delta/4$).

Fixing ρ and n we have $|B_n(z)| + |z| > \delta/2$ for every z , and $|B_n(z)| \geq \delta/4$ in R ; then $S_{\delta/2}(B_n) = \bigcup_{i \in I} D_i$, where each D_i is a connected component, which is simply connected by the maximum modulus principle; then there exists a logarithmic branch $\ln z$ on $D \setminus R$, such that, putting $\ln_i z = \ln z$ for $z \in D_i$ we have $|\ln_i z| \leq T_\delta = [(\ln \delta/4)^2 + (2\pi)^2]^{1/2}$ (observe that $|z| \geq \delta/4$ in $S_{\delta/2}(B_n)$), so by Lemma A, there exists $r_n \in H^\infty$ such that $r_n(a_k^i) = \ln_i a_k^i$ for every $a_k^i \in Z_{B_n} \cap D_i$ and $\|r_n\| \leq s = T_\delta(\delta/4)^{-\alpha}$, where α is an absolute constant. Then $z - e^{r_n}$ vanishes at every a_k^i , so there exists $h_n \in H^\infty$ such that $z - e^{r_n(z)} = h_n(z)B_n(z)$; then $|h_n(z)| = |z - e^{r_n(z)}|$ a.e. in S^1 in the sense of the nontangential limit function, and $\|h_n\| \leq 1 + e^{\|r_n\|} \leq 1 + e^s$. This shows that $\{r_n\}$ and $\{h_n\}$ are normal families (ρ remains fixed), and taking subsequences we have $h_n \rightarrow h_\rho$ and $r_n \rightarrow r_\rho$ uniformly on the compacts of D , for some h_ρ and r_ρ in H^∞ . Moreover, $\|h_\rho\| \leq 1 + e^s$, $\|r_\rho\| \leq s$ and $z - h_\rho(z)\phi_\rho(z)g_\rho(z) = e^{r_\rho(z)}$. By the definition of ϕ_ρ , $\|\phi_\rho\| \leq 3/\delta$. These three inequalities show that $\{\phi_\rho\}, \{h_\rho\}, \{r_\rho\}$ are normal families and, taking $\rho \rightarrow 1$ with $g_\rho(z) \neq 0$ in S^1 through a suitable sequence, we get $h_\rho \rightarrow h$, $\phi_\rho \rightarrow \phi$, $r_\rho \rightarrow r$ and $g_\rho \rightarrow g$ uniformly on the compacts of D for some h, ϕ, r in H^∞ with $\|h\| \leq 1 + e^s$, $\|r\| \leq s$, $\|\phi\| \leq 3/\delta$, and $z - h(z)\phi(z)g(z) = e^{r(z)}$. This shows that (z, g) is reducible, as desired.

REMARK. We have proved, indeed, that (f, g) is reducible to an element of H^∞ which is in the connected component of the neutral element.

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