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BAKER FUNCTIONS FOR COMPACT RIEMANN SURFACES

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ABSTRACT. This article contains a proof of an important theorem in soliton mathematics. The theorem, stated roughly in [4], contains necessary conditions for the existence of a vector function

$$\psi(t,p) = (\psi_1(t,p),\dots,\psi_l(t,p)), \qquad t \in \mathbf{C}^g, \qquad p \in R,$$

with prescribed poles and l essential singularities an a compact Riemann surface R of genus g. ψ is called a Baker function in honor of the 1928 article [1] of H. F. Baker. This report clarifies Krichever's description of ψ for $l > 1$ essential singularities. The divisors δ_α in (1) below are the key to the $l > 1$ construction. Krichever's ($l > 1$) construction is not easy to deal with in practical problems. E. Previato [5] noted this and applied our characterization of the δ_α to construct the finite gap solutions to the nonlinear Schroedinger equation.

1. Baker functions. Let there be given a compact Riemann surface R of genus g and fix $l(\geq 1)$ distinct points, $\infty = \infty_1 + \cdots + \infty_l$, of R. Let κ_α^{-1} be a local parameter vanishing at ∞_α and let Θ_α be a polynomial in κ_α. Let Δ be a positive nonspecial divisor on R. We define a complex linear space $\Lambda = \Lambda(\Delta, \infty, \Theta_\alpha)$ by the following properties. A function ψ belongs to Λ if

(Λ1) ψ is meromorphic in $p \in R - \infty$,
(Λ2) $(\psi) + \Delta \geq 0$, i.e., any pole of ψ lies in Δ,
(Λ3) near ∞_α, $\psi(p)e^{-\Theta_\alpha(\kappa_\alpha(p))}$ is holomorphic.

The purpose of this paper is to compute $\dim_{\mathbf{C}}(\Lambda)$. The idea is this. If the Θ_α are "small", in a sense to be made precise, then the zero divisor $\tilde{\Delta}$ of any function in Λ would be "close" to Δ. Since the nonspecial divisors are an open set, our choice of Δ would imply that $\tilde{\Delta}$ is a nonspecial divisor of degree equal to $\deg(\Delta)$. This, we will see in the proof of Theorem 3 below, implies

$$\dim_{\mathbf{C}}(\Lambda) = \deg(\Lambda) - g + 1.$$

The reader may consult [2] for a current article with background on these matters.

Let us introduce a vector $t = (t_0, t_1, \dots)$ of complex "time" parameters. Let there be given polynomials $\Theta_\alpha = \Theta_\alpha(t, \kappa_\alpha)$ with coefficients analytic in t and $\Theta_\alpha(0, \kappa_\alpha) = 0$. Let Δ be a positive nonspecial divisor of degree $g + l - 1$ such

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that $L(\Delta - \infty) = \{0\}$. The linear space $L(\Delta)$ contains a basis $f_1, \dots, f_l$ such that $f_\alpha(\infty_\beta) = \delta_{\alpha\beta}$. Each divisor

$$(1) \qquad \delta_\alpha \stackrel{\text{def}}{=} \Delta + (f_\alpha) - \infty + \infty_\alpha = (\Delta - (f_\alpha) - \infty) + ((f_\alpha)_0 - (\infty - \infty_\alpha))$$

$(\alpha = 1, \dots, l)$ is nonspecial and of degree g. The nonspeciality of δ_α is not obvious. To see this let $f \in L(\delta_\alpha)$. The product ff_α has poles in Δ and zeros at each point of $\infty - \infty_\alpha$. Therefore, $ff_\alpha = (\text{constant})f_\alpha$ and $f = \text{constant}$. This proves $L(\delta_\alpha) = \mathbf{C}$. It follows then that δ_α is nonspecial. The following theorem is a generalization of a construction apparently due to Clebsch and Gordan. See [**2**]. The notations of the proof follow [**3**].

2. PROPOSITION. *Let $(R, \infty, g, \kappa_\alpha, \Theta_\alpha)$ as above. Let δ be a nonspecial divisor of degree g and supported in $R - \infty$. Then there exists precisely one function $\chi = \chi_\delta$ with these properties:*

(2.1) *χ is single valued and meromorphic in $p \in R - \infty$, χ is analytic in t, $|t|$ sufficiently small,*

(2.2) *any pole of χ lies in δ,*

(2.3) *at ∞_α, $\chi(t,p)e^{-\Theta_\alpha(t,\kappa_\alpha(p))}$ is holomorphic,*

(2.3′) *at ∞_1, $\chi(t,p)e^{-\Theta_1(t,\kappa_1(p))} = 1 + O(\kappa_1^{-1})$.*

PROOF. Let us give a formula for a function χ, define the notation in the formula, verify that χ satisfies (2.1)–(2.3′), and then prove that any function satisfying (2.1)–(2.3) is a scalar times χ. Let

$$\chi(t,p) = C(t) f_t(p) f_0^{-1}(p) \exp\left(\sum_{\alpha=1}^{l} \int_{p_0}^{p} \omega_\alpha(t)\right)$$

where

$$f_t(p) = \Theta(\Phi(p) - \Phi(\delta) - K - B(\omega(t))/2\pi i),$$

$$C^{-1}(t) = e^{V_1(t)} f_t(\infty_1) f_0^{-1}(\infty_1) \exp \sum_{\alpha=2}^{l} \int_{p_0}^{\infty_1} \omega_\alpha(t).$$

We will show that $C(t)$ gives the normalization (2.3′) at ∞_1 and $f_t(\infty_1)$ is not zero for $|t|$ small.

NOTATION. The function $p \to \Phi(p)$ is the Abel map of R into the Jacobian variety J of R. To construct Φ and J let there be given a canonical homology basis $\alpha = (\alpha_1, \dots, \alpha_g)$, $\beta = (\beta_1, \dots, \beta_g)$, with intersection matrix $\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0$ and $\alpha_i \circ \beta_j = \delta_{ij}$. Let $\phi = (\phi_1, \dots, \phi_g)^T$ be the basis of holomorphic differentials with $\int_{\alpha_i} \phi_j = \delta_{ij}$. The $2g$ columns of the matrices $I = \int_\alpha \phi$ and $B = \int_\beta \phi$ of α and β periods of ϕ determine a lattice $L = \{m + Bn | m, n \in Z^g\}$ over the integers Z. The Jacobian variety of R is the compact, commutative, g-dimensional complex Lie group $J = \mathbf{C}^g/L$. Choose a point $p_0 \in R - \infty$. The Abel map is defined by $\Phi(p) = \int_{p_0}^{p} \phi$, and Φ is extended to divisors by $\Phi(p_1 + \cdots + p_m) = \sum_{j=1}^{m} \int_{p_0}^{p_j} \phi$.

The matrix B of β-periods of ϕ is symmetric with a positive definite imaginary part. This implies that the theta function

$$\Theta(z) = \sum_{n \in Z^g} \exp(2\pi i \langle z, n\rangle + \pi i \langle Bn, n\rangle)$$

is an entire function of $z \in \mathbf{C}^g$. One can verify by direct substitution that if B_j is column j of B and $m \in Z^g$ then $\Theta(z+m) = \Theta(z)$ and $\Theta(z+B_j) = \exp(-2\pi i z_j - \pi i B_{jj})\Theta(z)$.

For any $e \in \mathbf{C}^g$ the Riemann theta function $p \to \Theta(\Phi(p) - e)$ is either identically zero or it has precisely g zeros. If $p \to \Theta(\Phi(p) - e)$ is not identically zero and if $p_1 + \cdots + p_g$ is its zero divisor then $p_1 + \cdots + p_g$ is nonspecial and

$$\Phi(p_1 + \cdots + p_g) + K = e$$

(mod L) where $K = -\sum_{j=1}^{g} \int_{\alpha_j} \Phi \cdot \phi_j + \frac{1}{2}(B_{11}, \ldots, B_{gg})$. K is the vector of Riemann constants.

There exists a unique abelian differential $\omega_\alpha(t)$ of the second kind such that (i) $\omega_\alpha(t)$ has zero α-periods, (ii) the principal part of $\omega_\alpha(t)$ at ∞_α is $d\Theta_\alpha(t, k_\alpha)$ and (iii) ∞_α is the only pole of $\omega_\alpha(t)$. The $\omega_\alpha(t)$ vanish at $t = 0$ because $\Theta(0, \kappa_\alpha) = 0$ and the only abelian differential of the first kind with zero α-periods is identically zero. Let $\omega(t) = \sum_{\alpha=1}^{l} \omega_\alpha(t)$ and let $B(\omega(t)) = \int_\beta \omega(t)$ be the vector of β-periods of ω.

Consider the function $p \to \int_{p_0}^{p} \omega_1(t)$ restricted to a small neighborhood of ∞_1 and fix a path of integration. This function has the Laurent series expansion $\Theta_1(t, \kappa_1) + V_1(t) + O(\kappa_1^{-1})$ for some scalar $t \to V_1(t)$. For any path of integration,

$$\exp \int_{p_0}^{p} \omega_1(t) = e^{\Theta_1(t, \kappa_1(p))}(1 + O(\kappa_1^{-1}(p))) \exp(V_1(t) + j \cdot B(\omega_1(t)))$$

for some integral combination $j \cdot B(\omega_1(t))$ of β-periods of $\omega_1(t)$.

THE PROPERTIES OF χ. The argument of Θ in f_t depends analytically on t and it converges to that of f_0 as $t \to 0$. The zero divisor of f_0 is δ. The zero divisor $\delta(t)$ of f_t depends analytically on t and it moves to δ as $t \to 0$. The nonspecial divisors are open in the space of divisors; hence, $\delta(t)$ will be nonspecial for $|t|$ sufficiently small. Since $\infty_1 \notin \delta$ and because ∞_1 is not a pole of the $\omega_\alpha(t)$ $(\alpha \geq 2)$, the scalar $C(t)$ is finite and nonzero. The normalization (2.3′) of χ follows from the definition of $C(t)$ and $V_1(t)$. Let us show that χ is single valued.

The dissected form of R is a $4g$ sided polygon $\prod_{j=1}^{g} \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1}$ with the edge α_j (β_j) identified with the edge of α_j^{-1} (β_j^{-1}). Let $q \in \alpha_j$ be identified with $\overline{q} \in \alpha_j^{-1}$ and let $p \in \beta_j$ be identified with $\overline{p} \in \beta_j^{-1}$. The identities ([**3**, p. 285, formulas (1.5)] with $\varepsilon = \varepsilon' = 0$)

$$f_t(\overline{p}) = f_t(p),$$
$$f_t(\overline{q}) = f_t(q) \exp(-\pi i B_{jj} - 2\pi i(\Phi_j(q) - \Phi_j(\delta) - K_j))e^{-B_j(\omega(t))},$$
$$\int_{\overline{q}}^{q} \omega = -\sum_{\alpha=1}^{l} B_j(\omega_\alpha(t)) = -B_j(\omega(t)), \qquad \int_{\overline{p}}^{p} \omega = \sum_{\alpha=1}^{l} \int_{\alpha_j} \omega_\alpha(t) = 0,$$

imply that χ is single valued.

UNIQUENESS OF χ. Suppose $\hat{\chi}(t,p)$ is another function with properties (2.1)–(2.3). The quotient $\sigma = \hat{\chi}(t,p)\chi^{-1}(t,p)$ is a meromorphic function of p. The divisor of poles of σ is contained in the (zero) divisor of f_t; it is nonspecial for $|t|$ small. The only function with poles in a nonspecial divisor of degree g is constant. Thus $\sigma = \sigma(t)$. □

The following theorem reducesd to (2.2) when $l = 1$. It is the main result of this paper. There are two keys to its proof: the divisors δ_α defined in (2.1) and a nice generalization of the uniqueness argument in (2.2) due to the referee.

3. THEOREM. *Let R be a Riemann surface of genus g. Fix l points $\infty = \infty_1 + \cdots + \infty_l$. Let Δ be a positive nonspecial divisor of degree $g + l - 1$ such that $L(\Delta - \infty) = \{0\}$. Choose a local parameter z for a neighborhood of ∞_α such that $z_\alpha(\infty_\alpha) = 0$ and let $\kappa_\alpha = z_\alpha^{-1}$. Choose polynomials $\Theta_\alpha = \Theta_\alpha(t, \kappa_\alpha)$ in κ_α and analytic in t such that $\Theta_\alpha(0, \kappa_\alpha) = 0$. The set $\Lambda = \Lambda(\Delta, \infty, \Theta_\alpha)$ of functions $\chi(t,p)$ such that*

(3.1) *χ is meromorphic in $R - \infty$ and analytic in t, $|t|$ small,*

(3.2) *any pole of χ lies in Δ,*

(3.3) *near ∞_α, $\chi(t,p)e^{-\Theta_\alpha(t,\kappa_\alpha(p))}$ is holomorphic,*

is a complex linear space of dimension l. Moreover, there is a basis ψ_α for Λ such that, near ∞_β,

(3.3′) *$\psi_\alpha(t,p)e^{-\Theta_\beta(t,\kappa_\beta(p))} = \delta_{\alpha\beta} + O(\kappa_\beta^{-1})$.*

PROOF. We apply Proposition (2.2) with $\delta = \delta_\alpha$ in (1) to obtain a function χ_α with divisor of poles δ_α and normalized at ∞_α. The function

$$\psi_\alpha(t,p) \stackrel{\text{def}}{=} f_\alpha(p)\chi_\alpha(t,p)$$

belongs to Λ and it satisfies (3.3′). It is clear that the ψ_α ($\alpha = 1, \ldots, l$) are linearly independent and therefore $\dim(\Lambda) \geq l$. We have to show that $\dim(\Lambda)$ is not greater than l. Let ψ be any element of Λ. There exists a positive divisor $\Delta_\psi(t)$ such that $(\psi(t,\cdot)) = \Delta_\psi(t) - \Delta$. We have $\Delta_\psi(t) = (\psi(t,\cdot))_0$ unless a zero of ψ cancels a point of Δ at some instant in time. In the latter case we have $\Lambda_\psi(t) = (\psi(t,\cdot))_0+$ (cancelled points of Δ). We have

$$\begin{aligned}
(\psi_\alpha) &= (f_\alpha) + (\chi_\alpha) = (f_\alpha) + (\chi_\alpha)_0 - \delta_\alpha \\
&= (f_\alpha) + (\chi_\alpha)_0 - (\Delta + (f_\alpha) - \infty + \infty_\alpha) \quad \text{(by (1))} \\
&= (\chi_\alpha)_0 + (\infty - \infty_\alpha) - \Delta, \\
(\psi/\psi_1) &= (\psi) - (\psi_1) = \Delta_\psi(t) - ((\chi_1)_0 + (\infty - \infty_\alpha)) \quad \text{and} \\
(\psi_\alpha/\psi_1) &= (\psi_\alpha) - (\psi_1) = (\chi_\alpha)_0 + \infty_1 - ((\chi_1)_0 + \infty_\alpha).
\end{aligned}$$

These formulas show that any pole of the $l + 1$ *meromorphic* functions ψ/ψ_1, $\psi_1/\psi_1 = 1$, $\psi_2/\psi_1, \ldots, \psi_l/\psi_1$ lies in $(\chi_\alpha)_0 + \infty - \infty_1$, a nonspecial divisor of degree $g + l - 1$. The $l + 1$ functions are, by the Riemann-Roch theorem, linearly dependent. □

Note. The referee has informed me of the following characterization of the important divisor δ_α. It is the unique integral divisor of degree g such that

$$\Phi(\Delta) - \Phi(\infty - \infty_\alpha) = \Phi(\delta_\alpha).$$

Note. Let δ be an integral nonspecial divisor of degree g supported in $R - \infty$ and let $p_1, \ldots, p_{l-1}$ be distinct points in $R - \infty$. By the Riemann-Roch there exists a nonconstant function $g_j \in L(\delta + p_j)$ such that $g_j(\infty_j) = 0$. Since δ is nonspecial, g_j has a pole at p_j and it is unique up to a constant factor. Let $g_l = 1$. The l functions $g_1, \ldots, g_l$ are a basis for $L(\delta + p_1 + \cdots + p_{l-1})$ and the divisor

$$\Delta \stackrel{\text{def}}{=} \delta + (p_1 + \cdots + p_{l-1})$$

satisfies the hypothesis of Theorem 3 (i.e. $L(\Delta) = \{0\}$) if and only if the determinant of the $l \times l$ matrix $(g_j(\infty_k))$ is nonzero. Thus we have a way of searching for divisors Δ satisfying $L(\Delta - \infty) = 0$. The method is probably not practical unless R is hyperelliptic because our explicit formulas for g_j are quite unmanageable even for a trigonal R.

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