**LOCALLY COMPACT NORMAL METALINDELÖF SPACES
MAY NOT BE PARACOMPACT:
AN APPLICATION OF UNIFORMIZATION AND SUSLIN LINES**

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ABSTRACT. We show that it is consistent with and independent of the continuum hypothesis (and its negation) that there is a locally compact (perfectly) normal metalindelöf space which is not paracompact. The constructions replace each point in an ω -uniformizable family with a Suslin line.

0. Introduction. A problem of Arhangel'skiĭ and Tall which has been much worked upon in the last five years is: Are locally compact normal metacompact spaces paracompact?

In 1959, Arhangel'skiĭ [1] showed that locally compact perfectly normal metacompact spaces are paracompact. In 1974, Tall discussed the question in [2]. In 1980, the author [3] showed that, under $V = L$, locally compact normal metacompact spaces are paracompact. In 1982, Daniels [4] showed that locally compact normal boundedly metacompact spaces are paracompact.

This was a surprising result which raised the possibility of a theorem in ZFC alone. In 1984, Balogh [6] showed that, under $V = L$, locally compact normal metalindelöf spaces are paracompact. This was another surprising result which raised the possibility of a theorem in ZFC which did not even use the finitistic methods of Daniels' proof. In this paper, we shed more light on the question by showing that ZFC does not imply that locally compact normal metalindelöf spaces are paracompact. This narrows the search for a solution to Arhangel'skiĭ's problem. Either an upgrading of Daniels' proof (a difficult task) will provide a solution in ZFC or perhaps a MA-style counterexample can be devised which is not boundedly metacompact (difficult to imagine!).

1. The construction. The idea of the construction is to take standard examples of locally compact normal spaces which are not collectionwise Hausdorff and split points. We must split the points enough to get metalindelöf but not so much as to get metacompact. This is not difficult but we must do it inside a compact space to keep the local compactness.

Let $E \in [\omega_1]^{\omega_1}$. Let $\eta = \{\eta_\alpha : \alpha \in E\}$ be an almost disjoint family of countable subsets of $\omega_1 - E$.

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DEFINITION. η is ω -uniformizable if, whenever $\chi_\alpha: \eta_\alpha \rightarrow \omega$ for each $\alpha \in E$ (χ_α colors η_α), there is $\chi: \omega_1 - E \rightarrow \omega$ (χ uniformizes the colorings) such that χ almost contains χ_α (mod finite).

DEFINITION. η is *almost point-countable* if there are $\{F_\alpha: \alpha \in E\} \subset [\omega_1 - E]^{<\omega}$ such that $\{\eta_\alpha - F_\alpha: \alpha \in E\}$ is point-countable.

We shall show

THEOREM 1. *If there is an ω -uniformizable family of countable sets which is not almost point-countable and if there is a compact 0-dimensional L -space then there is a metalindelöf locally compact (perfectly) normal space which is not paracompact.*

PROOF. Let $E \in [\omega_1]^{\omega_1}$ and let $\{\eta_\alpha: \alpha \in E\}$ be an ω -uniformizable not almost point-countable almost disjoint family of countable subsets of $\omega_1 - E$ (where each η_α is listed by $\{\eta_\alpha(n): n \in \omega\}$). Let L be a compact 0-dimensional L -space. The idea is to split each point of $\omega_1 - E$ into a copy of L . Šapировskii [8] has shown that the density of a hereditarily Lindelöf space is at most $\aleph_1$ so, letting $\{d_\alpha: \alpha \in \omega_1\}$ be dense in L , we can let L_α be the closure of $\{d_\beta: \beta < \alpha\}$ for each $\alpha \in \omega_1$. $\{L_\alpha: \alpha \in \omega_1\}$ is an increasing sequence of closed sets and $\bigcup\{L_\alpha: \alpha \in \omega_1\} = L$. Choose a clopen $K_\alpha \subset L - L_\alpha$. $\{K_\alpha: \alpha \in \omega_1\}$ is a point-countable family of clopen sets. The idea is for each $\alpha \in E$ to "look down" at the copies of K_α .

Let $X = E \cup ((\omega_1 - E) \times L)$. X is topologized by letting $X - E$ be open and have the product topology (where $\omega_1 - E$ has the discrete topology) and for each $\alpha \in E$, letting $U_n(\alpha) = \{\alpha\} \cup (\{\eta_\alpha(m): m > n\} \times K_\alpha)$ be open for each $n \in \omega$.

X is a 0-dimensional Hausdorff space. X metalind because $\{U_0(\alpha): \alpha \in E\}$ is a point-countable family (if $(b, d) \in X - E$ lies in uncountably many $U_0(\alpha)$ then d lies in uncountably many K_α) and because only finitely many open subsets of $\{\alpha\} \times L$ are needed to cover $\{\alpha\} \times L$. X is locally compact because $U_0(\alpha)$ is compact. (L is compact open and $U_0(\alpha) - U_n(\alpha)$ is contained in the union of finitely many copies of L). We show that X is normal in two steps by applying a lemma:

LEMMA 1. *If a Hausdorff space X is the union of two subspaces A, B such that*

(1) *A is closed, discrete and normalized,*

(2) *B is normal,*

(3) *for each open $U \supset A$ there is a clopen V such that $U \supset V \supset A$,*

then X is normal.

PROOF. Let H, K be disjoint closed sets in X . Assume $H \cup K \supset A$. $H \cap A$ and $K \cap A$ are disjoint closed sets. By (1), let G and J be disjoint open sets such that $G \supset H \cap A$ and $J \supset K \cap A$. Assume $G \cap K = J \cap H = \emptyset$. Let $U = G \cup J$. By (3), let V be clopen such that $U \supset V \supset A$. $V \cap G$ and $V \cap J$ are disjoint clopen sets (suppose $V \cap G$ has a limit point p ; $p \notin J$ since J is open, $p \notin X - V$ since $X - V$ is open, $p \in V - J \subset U - J = G$). $H - V$ and $K - V$ are disjoint closed sets in the clopen subspace $X - V$. $X - V \subset B$ implies by (2) that $X - V$ is normal. Let M, N be disjoint open sets in $X - V$ such that $M \supset H - V$ and $N \supset K - V$. $M \cup (V \cap G)$ and $N \cup (V \cap J)$ are disjoint open sets separating H and K ($M \cup (V \cap G) \supset H - V$ and $M \cup (V \cap G) \supset V \cap G \cap H = V \cap H$).

PROOF OF THEOREM 1 (CONTINUED). In the first step, we use 2-uniformizability of the ladder system to show that E is normalized. This is a standard argument. Let H and K partition E . For each $\alpha \in E$, let $\chi_\alpha: \eta_\alpha \rightarrow 2$ (coloring η_α) be defined by $\chi_\alpha(\beta) = 0$ iff $\alpha \in H$. Let $\chi: \omega_1 - E \rightarrow 2$ (uniformizing the colorings) be such that each χ almost contains χ_α (mod finite). Let $U = H \cup \bigcup\{\{\alpha\} \times L: \chi(\alpha) = 0\}$ and $V = K \cup \bigcup\{\{\alpha\} \times L: \chi(\alpha) = 1\}$. U and V are disjoint open subset of χ , e.g. if $\alpha \in H$ and n is such that $\chi(\eta_\alpha(m)) = \chi_\alpha(\eta_\alpha(m)) = 0$ for $m > n$ then $U_n(\alpha) \subset U$. In the second step, we use ω -uniformizability of the ladder system to show that E can be separated from any closed subset of $X - E$. Let $U \supset E$ be open. We shall find a clopen V such that $U \supset V \supset E$. For each $\alpha \in \omega_1 - E$, let $(\{\alpha\} \times L) \cap U = \bigcup\{F_\alpha^n: n > 0\}$ where each F_α^n is clopen and each $F_\alpha^n \subset F_\alpha^{n+1}$ (this is possible by perfect normality and strong 0-dimensionality; F_α^n can be empty). For each $\alpha \in E$, let $\chi_\alpha: \eta_\alpha \rightarrow \omega$ (coloring the η_α) be defined by $\chi_\alpha(\beta) = \inf\{m: \{\beta\} \times K_\alpha \subset F_\beta^m\}$ (letting the infimum of $\emptyset$ be 0). Let $\chi: (\omega_1 - E) \rightarrow \omega$ (uniformizing the colorings) be such that, for each $\alpha \in E$, χ almost contains χ_α (mod finite). For each $\alpha \in E$ let $n(\alpha)$ be such that $\chi \supset \chi_\alpha \upharpoonright \{\eta_\alpha(n): n > n(\alpha)\}$ and such that $n > n(\alpha)$ implies $\chi_\alpha(\eta_\alpha(n)) > 0$. This is possible since there is $n(\alpha) \in \omega$ such that $n > n(\alpha)$ implies $U_n(\alpha) \subset U$ and $\{\eta_\alpha(n)\} \times K_\alpha \subset (\{\eta_\alpha(n)\} \times L) \cap U$ and so, for some $m \in \omega$, $\{\eta_\alpha(n)\} \times K_\alpha \subset F_{\eta_\alpha(n)}^m$.

Let $V = \bigcup\{F_\alpha^{X(\alpha)}: \alpha \in \omega_1 - E\} \cup E$. Each $F_\alpha^{X(\alpha)} \subset U$ so $V \subset U$. V is closed since $V \supset E$ and $V \cap (\{\alpha\} \times L) = F_\alpha^{X(\alpha)}$ is clopen. V is open since each $U_{n(\alpha)}(\alpha) \subset V$. Let $n > n(\alpha)$. Let $\beta = \eta_\alpha(n)$. We show that $\{\beta\} \times K_\alpha \subset F_\beta^{X(\beta)}$. $\chi_\alpha(\beta) > 0$ while $\chi(\beta) = \chi_0(\beta)$ implies $\{\beta\} \times K_\alpha \subset F_\beta^{X(\beta)}$. We show that X is not paracompact by showing that E is unseparated. (Balogh [6] showed that locally compact normal metalindelöf collectionwise Hausdorff spaces are paracompact.) Suppose $\{U_{n(\alpha)}(\alpha): \alpha \in E\}$ is a disjoint family of open sets. $\{\eta_\alpha: \alpha \in E\}$ is not almost point-countable implies that $\{\{\eta_\alpha(m): m > n(\alpha)\}: \alpha \in E\}$ is not point-countable. Let $F \in [E]^{\omega_1}$ and $\beta \in \{\eta_\alpha(m): m > n(\alpha)\}$ for each $\alpha \in F$. $\{K_\alpha: \alpha \in F\}$ is not a disjoint family of open subsets of L , so let $\alpha, \alpha' \in F$ and $d \in L$ be such that $d \in K_\alpha \cap K_{\alpha'}$. $(\beta, d) \in U_{n(\alpha)}(\alpha) \cap U_{n(\alpha')}(\alpha')$.

We show that X is perfect with

LEMMA 2. *If a Hausdorff space X is the union of two subspaces A and B such that (1) A is closed discrete and a G_δ and (2) B is perfect, as a subspace, then X is perfect.*

PROOF. Let H be a closed set in X . Let $\{U_n: n \in \omega\}$ be a decreasing sequence of open sets in X such that $\bigcap\{U_n: n \in \omega\} = A$. Let $\{V_n: n \in \omega\}$ be a decreasing sequence of open sets in $X - A$ such that $\bigcap\{V_n: n \in \omega\} = H - A$. Any element of $(X - (A - H)) \cap \bigcap\{U_n \cup V_n: n \in \omega\}$ is either in infinitely many U_n and thus in A and thus in $A \cap H$ or is in infinitely many V_n and thus in $H - A$.

PROOF OF THEOREM 1 (CONTINUED). By Lemma 2, we need only show that E is a G_δ (the free union of hereditary lindelöf sets is perfect). We use the ω -uniformizability of η again. Let each χ_α be any one-to-one function. Let $U_n = E \cup \bigcup\{\{\beta\} \times L: \chi(\beta) > n\}$ where χ uniformizes the χ_α . U_n is open since $\alpha \in E$ implies $U_m(\alpha) \subset U_n$ whenever $p > m$ implies $\chi(\eta_\alpha(p)) > n$.

2. The examples. We need a lemma and a definition.

THEOREM 3. *If there is a Suslin line X , then there is a compact 0-dimensional L -space.*

PROOF. If $(X, <)$ is a compact Suslin line, then let $X \times 2$ have the lexicographic ordering and the order topology. $X \times 2$ is a compact 0-dimensional L -space.

DEFINITION. If $E \in [\omega_1]^{\omega_1}$ consists of limit ordinals and $\eta = \{\eta_\alpha: \alpha \in E\}$ is a family of subsets of $\omega_1 - E$ such that α is the unique limit point of η_α , then η is a *ladder system*.

Note that a ladder system is an almost disjoint family and that if η is a ladder system and E is stationary, then η is not almost point-countable.

THEOREM 2 (BALOGH [6]). *$V = L$ implies that locally compact metalindelöf normal spaces are paracompact.*

THEOREM 3. *$\text{Con}(\text{ZF})$ implies $\text{Con}(\text{ZFC} + \diamond + \text{there is a locally compact metalindelöf (perfectly) normal space which is not paracompact})$.*

PROOF. Shelah [5] has constructed a model of $\diamond$ in which there is a stationary set E and a ω -uniformizable ladder system $\eta = \{\eta_\alpha: \alpha \in E\}$. $\diamond$ implies that there is a Suslin line. These theorems show that it is consistent with and independent of $\diamond$ that locally compact metalindelöf (perfectly) normal spaces be paracompact.

THEOREM 4. *$\text{Con}(\text{ZF})$ implies $\text{Con}(\text{ZFC} + 2^{\aleph_0} = \aleph_2 + \text{locally compact metalindelöf normal spaces are paracompact})$.*

PROOF. Use Balogh's result in [6] that locally compact metalindelöf normal collectionwise Hausdorff spaces are paracompact and Tall's result in [9] that, in the model obtained by adding $\aleph_2$ random reals to L , locally compact normal spaces are collectionwise Hausdorff.

THEOREM 5. *$\text{Con}(\text{ZF})$ implies $\text{Con}(\text{ZFC} + 2^{\aleph_0} = \aleph_2 + p = c + \text{there is a locally compact metalindelöf (perfectly) normal space which is not paracompact})$.*

PROOF. By [10], there is a model of $\text{MA}_{\aleph_1}$ for property K partial orders in which there is a Suslin line. The forcing partial order for showing that any ladder system is ω -uniformizable under $\text{MA}_{\aleph_1}$ [7] has property K . This construction cannot be made Moore because locally compact metalindelöf Moore spaces are metrizable.

These theorems show that it is consistent with and independent of $\neg\text{CH}$ that locally compact metalindelöf (perfectly) normal spaces be paracompact.

THEOREM 6 (TALL [11]). *$\text{MA}_{\aleph_1}$ implies locally compact metalindelöf c.c.c. spaces are paracompact.*

THEOREM 7. *$\text{Con}(\text{ZF})$ implies $\text{Con}(\text{ZFC} + 2^{\aleph_0} = \aleph_2 + \text{there is a locally compact metalindelöf (perfectly normal) c.c.c. space which is not paracompact (equivalently, Lindelöf)})$.*

PROOF (SEE II.4.6 OF [7]). There is a model in which there is an ω -uniformizable family η such that $\bigcup \eta$ is countable (thus implying that η is not almost point-countable). This model is obtained by property K forcing over any model of set theory. If the ground model has a Suslin line, then so does the extension.

These theorems show that it is consistent with and independent of $\neg\text{CH}$ that locally compact metalindelöf c.c.c. (perfectly normal) spaces are lindelöf.

CONJECTURE. $\text{MA}_{\aleph_1}$ implies that normal metalindelöf locally compact spaces are paracompact.

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