CONVOLUTION ESTIMATES FOR SOME MEASURES ON CURVES

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ABSTRACT. Suppose that λ is a smooth measure on a curve in R^3 . It is shown that $\lambda * L^p(R^3) \subset L^q(R^3)$ under certain conditions on λ, p , and q .

For $1 \leq p \leq \infty$, let L^p be the usual Lebesgue space formed with respect to Lebesgue measure on R^n . It is well known that every complex Borel measure λ on R^n acts as a convolution operator on L^p : $\lambda * L^p \subset L^p$. If λ is absolutely continuous with density in L^r for some $r > 1$, Young's inequality shows that for $1 \leq p < \infty$ we have $\lambda * L^p \subset L^q$ for some $q = q(p) > p$. One might say that such a measure is " L^p -improving." If λ is singular, it is still possible for λ to be L^p -improving when $p > 1$. The Cantor-Lebesgue measures on R are examples [1]. The purpose of this paper is to investigate similar behavior for a different class of singular measures, smooth measures supported on curves in R^3 . Our motivation comes from the following two results.

THEOREM 1. *Suppose $1 \leq k < n$ and K is a smooth k -dimensional surface in R^n . Suppose that λ is a smooth measure on K . If $\lambda * L^p \subset L^q$, then $(1/p, 1/q)$ lies in the closed triangle in R^2 with vertices $(0, 0)$, $(1, 1)$, and $(n/(2n-k), (n-k)/(2n-k))$.*

PROOF. A theorem of Hörmander [3] implies that $p \leq q$. Estimating the norms of f and $\lambda * f$ when f is the characteristic function of a small ball shows that $(1/p, 1/q)$ lies on or above the line L joining $(1, 1)$ and $(n/(2n-k), (n-k)/(2n-k))$. Since $\lambda * L^p \subset L^q$ implies $\lambda * L^{q'} \subset L^{p'}$, where p' and q' are the conjugate indices, $(1/q', 1/p')$ also lies on or above L . This completes the proof.

THEOREM 2. *Suppose K is a smooth $(n-1)$ -dimensional surface in R^n on which all $n-1$ principal curvatures are bounded away from zero. Suppose that λ is a smooth finite measure compactly supported away from the boundary. Then $\lambda * L^p \subset L^q$ if and only if $(1/p, 1/q)$ lies in the closed triangle in R^2 with vertices $(0, 0)$, $(1, 1)$, and $(n/(n+1), 1/(n+1))$.*

PROOF. A theorem of Littman [4] implies that $\lambda * L^{(n+1)/n} \subset L^{n+1}$. Interpolation with trivial cases completes half the proof, and Theorem 1 gives the rest.

Theorems 1 and 2 lead naturally to the conjecture that if λ is a nice measure supported on a nice k -dimensional surface in R^n , then $\lambda * L^p \subset L^q$ precisely when $(1/p, 1/q)$ is in the closed triangle of Theorem 1. This conjecture seems difficult even when $k = 1, n = 3$. Our purpose here is to prove a partial result for this case. We consider certain measures λ on some curves in R^3 and prove that $\lambda * L^p \subset L^q$ whenever $(1/p, 1/q)$ lies in the triangle of Theorem 1 and on or above the line

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through $(1/2, 1/3)$ and $(2/3, 1/2)$. To show this it is sufficient, by duality and interpolation, to establish that $\lambda * L^{3/2} \subset L^2$.

THEOREM 3. *Suppose that I is a closed interval in R and that the real-valued functions ϕ_1 and ϕ_2 are both polynomial functions or both trigonometric functions on I . Put $\phi(t) = (\phi_1(t), \phi_2(t))$ and, if $j = 2$ or 3 , write $\phi^{(j)}$ for the j th derivative of the function ϕ . Suppose that given any $t_1, t_2 \in I$, the vectors $\phi^{(2)}(t_1)$ and $\phi^{(3)}(t_2)$ span R^2 . Let λ be the measure on R^3 defined by*

$$\langle \lambda, g \rangle = \int_I g(t, \phi_1(t), \phi_2(t)) dt.$$

Then $\lambda * L^{3/2} \subset L^2$.

Here are two examples where Theorem 3 applies.

EXAMPLE 1. $I = [a, b]$, $\phi_1(t) = t^2$, $\phi_2(t) = t^3$.

EXAMPLE 2. $I = [0, \pi/6]$, $\phi_1(t) = \cos(t)$, $\phi_2(t) = \sin(t)$.

Following are some comments on the notation to be used in the proof of Theorem 3. The symbol f (resp. g) will denote an arbitrary continuous function of compact support on R^2 (resp. R^3). The symbol $\|\cdot\|_p$ will denote the norm of the indicated function in $L^p(R^2)$ or $L^p(R^3)$, whichever is appropriate. A sector Γ in the plane is defined to be the set of all points in the plane which have a polar representation $re^{i\theta}$ with $r \geq 0$, $c \leq \theta \leq d$, where c and d are fixed real numbers. The symbol C will denote a positive constant which may increase from line to line but which depends only on the data I and ϕ in the hypotheses of Theorem 3. Similarly, the symbol $C(\Gamma_1, \Gamma_2, \delta)$ appearing in the following lemma represents a "variable" constant which can always be chosen to depend only upon Γ_1, Γ_2 , and δ . The proof of this lemma is based on complex interpolation and follows fairly standard lines.

LEMMA. *Suppose Γ_1 and Γ_2 are sectors such that $\Gamma_2 \cap (\Gamma_1 \cup -\Gamma_1) = \{0\}$. Suppose $\delta > 0$. There is a positive constant $C(\Gamma_1, \Gamma_2, \delta)$ such that the following holds: Suppose $J \subset R$ is a closed interval of length not exceeding δ^{-1} and suppose that $\Psi(t) = (\Psi_1(t), \Psi_2(t))$ is a twice differentiable function from J into R^2 such that*

- (i) *for every $t \in J$: $\Psi'(t) \in \Gamma_1$, $\Psi^{(2)}(t) \in \Gamma_2$, $|\Psi'(t)| \geq \delta$, $|\Psi^{(2)}(t)| \geq \delta$;*
- (ii) *for any $x \in R^2$, J splits into disjoint subintervals $J_1, \dots, J_K$ with $K \leq \delta^{-1}$ such that the scalar product $x \cdot \Psi^{(2)}(t)$ is of constant sign on each t -interval J_n .*

The measure μ on R^2 defined by

$$\langle \mu, f \rangle = \int_J f(\Psi(t)) dt$$

satisfies

$$\|\mu * f\|_3 \leq C(\Gamma_1, \Gamma_2, \delta) \|f\|_{3/2}.$$

PROOF. For $z \in C$ consider the distribution $d_z(s) = |s|^z$ on R . The map $z \rightarrow d_z$ defines a meromorphic distribution-valued function on C with simple poles at $z = -1, -3, -5, \dots$. Thus $d_z/(\Gamma(z+1)/2)$ is an entire distribution-valued function of z (see [2] for details).

Now let y_1 be a unit vector orthogonal to no nonzero vector in Γ_1 and let y be a unit vector orthogonal to y_1 .

For $z \in C$ and test functions h on R^2 , define the distribution T_z by

$$\langle T_z, h \rangle = \frac{1}{\Gamma((z+1)/2)} \int_J \int_{-\infty}^{\infty} h(\Psi(t) + sy) |s|^z ds dt.$$

We will establish that

$$(1) \quad \|T_{iy} * h\|_{\infty} \leq \left(C(\Gamma_1, \Gamma_2, \delta) / \Gamma\left(\frac{1+iy}{2}\right) \right) \|h\|_1,$$

$$(2) \quad \|T_{iy-3/2} * h\|_2 \leq \left(C(\Gamma_1, \Gamma_2, \delta) / \Gamma\left(\frac{3-2iy}{4}\right) \right) \|h\|_2,$$

$$(3) \quad T_{-1} * h = \mu * h.$$

Then the conclusion of the lemma will follow from the interpolation theorem in [5].

To show (1) is to show that T_{iy} is an L^{∞} function of appropriate norm. The map $(s, t) \rightarrow \Psi(t) + sy$ from $R \times J$ into R^2 is one-to-one by the assumptions on y and y_1 and the mean value theorem. It has Jacobian of absolute value $|\Psi'(t) \cdot y_1|$, a quantity which exceeds $1/C(\Gamma_1, \Gamma_2, \delta)$. Thus

$$\left| \int_J \int_{-\infty}^{\infty} h(\Psi(t) + sy) ds dt \right| \leq C(\Gamma_1, \Gamma_2, \delta) \|h\|_1.$$

This gives (1).

To obtain (2) we must estimate the Fourier transform of $T_{iy-3/2}$. Calculations in [2] show that the Fourier transform of $d_z/\Gamma((z+1)/2)$ at s is

$$2^{z+1} \pi^{1/2} |s|^{-z-1} / \Gamma(-z/2).$$

Thus the Fourier transform of T_z is given by

$$\hat{T}_z(x) = \hat{\mu}(x) \cdot 2^{z+1} \pi^{1/2} |x \cdot y|^{-z-1} / \Gamma(-z/2)$$

(from which (3) follows). To establish (2) it is therefore sufficient to show that

$$(4) \quad |x|^{1/2} |\hat{\mu}(x)| \leq C(\Gamma_1, \Gamma_2, \delta) \quad \text{for } x \in R^2.$$

Note that the hypotheses guarantee that

$$(5) \quad |x \cdot \Psi'(t)| + |x \cdot \Psi^{(2)}(t)| \geq |x|/C(\Gamma_1, \Gamma_2, \delta), \quad x \in R^2, t \in J.$$

Now fix nonzero $x \in R^2$ and put $p(t) = x \cdot \Psi(t)$. By (ii), J splits into disjoint subintervals $J_1, \dots, J_K$ with $K \leq \delta^{-1}$ such that $p^{(2)}(t)$ has constant sign on each J_n . Write $\eta = |x|/2C(\Gamma_1, \Gamma_2, \delta)$ and let $I_n^1 = J_n \cap \{|p'(t)| < \eta\}$. Since $p'(t)$ is monotone on J_n , I_n^1 is an interval and so $J_n \setminus I_n^1$ is the disjoint union of I_n^2 and I_n^3 , where each of I_n^2 and I_n^3 is an interval or empty. By (5) we have either $p^{(2)}(t) \geq \eta$ or $p^{(2)}(t) \leq -\eta$ on I_n^1 . Therefore

$$(6) \quad \eta^{1/2} \left| \int_{I_n^1} e^{-ix \cdot \Psi(t)} dt \right| \leq 4$$

by van der Corput's lemma [7]. If $j = 2$ or $j = 3$ we have $p'(t)$ monotone and either $p'(t) > \eta$ or $p'(t) < -\eta$ on I_n^j . Thus, by van der Corput's lemma again,

$$(7) \quad \eta^{1/2} \left| \int_{I_n^j} e^{-ix \cdot \Psi(t)} dt \right| \leq \min\{\eta^{1/2} \cdot \text{length}(I_n^j), \eta^{-1/2}\} \leq \delta^{-1/2},$$

where the last inequality is because $\text{length}(J) \leq \delta^{-1}$. Adding all inequalities (6) and (7) gives (4) since there are at most $3K \leq 3\delta^{-1}$ such terms. This completes the proof of the lemma.

PROOF OF THEOREM 3. Let $\tilde{\lambda}$ be defined by

$$\langle \tilde{\lambda}, g \rangle = \int_{R^3} g(-x) d\lambda(x).$$

Since $\langle \tilde{\lambda} * \lambda * g, g \rangle = \|\lambda * g\|_2^2$, it is enough to show that

$$(8) \quad \|\tilde{\lambda} * \lambda * g\|_3 \leq C\|g\|_{3/2}$$

for continuous compactly supported functions g on R^3 . Writing $I = [a, b]$ and $I_u = [\max\{a, a-u\}, \min\{b, b-u\}]$ for $|u| \leq b-a$ we have

$$\begin{aligned} \langle \tilde{\lambda} * \lambda, g \rangle &= \int_a^b \int_a^b g(t-s, \phi_1(t) - \phi_1(s), \phi_2(t) - \phi_2(s)) ds dt \\ &= \int_{a-b}^{b-a} \int_{I_u} g(u, \phi_1(s+u) - \phi_1(s), \phi_2(s+u) - \phi_2(s)) ds du. \end{aligned}$$

For $|u| \leq b-a$, define the measure λ_u on R^2 by

$$\langle \lambda_u, f \rangle = \int_{I_u} f(\phi_1(s+u) - \phi_1(s), \phi_2(s+u) - \phi_2(s)) ds.$$

We will prove that

$$(9) \quad \|\lambda_u * f\|_3 \leq C|u|^{-2/3}\|f\|_{3/2}.$$

Assume (9) for the moment and write $g(t, x)$ ($t \in R$, $x \in R^2$) for a continuous compactly supported function on R^3 . Then

$$\begin{aligned} \|\tilde{\lambda} * \lambda * g\|_3 &= \left\| \left\| \int_{a-b}^{b-a} \lambda_u * g(t-u, \cdot)(x) du \right\|_{3,x} \right\|_{3,t} \\ &\leq \left\| \int_{a-b}^{b-a} \|\lambda_u * g(t-u, \cdot)(x)\|_{3,x} du \right\|_{3,t} \\ &\leq C \left\| \int_{a-b}^{b-a} |u|^{-2/3} \|g(t-u, x)\|_{3/2,x} du \right\|_{3,t}. \end{aligned}$$

By the boundedness of the Riesz potential of order $\frac{1}{3}$ as a mapping from $L^{3/2}(R)$ to $L^3(R)$ (see p. 119 of [6]), this last term is dominated by

$$C\|g(t, x)\|_{3/2,x}\|_{3/2,t} = C\|g\|_{3/2}.$$

Thus (8), and so Theorem 3, will be proved when (9) is established.

By the hypotheses of Theorem 3 there is an $\eta > 0$ such that

$$(10) \quad |\phi^{(2)}(t)|, |\phi^{(3)}(t)| \geq \eta \quad \text{if } t \in I.$$

The sets

$$K_1 = \{\phi^{(2)}(t)/|\phi^{(2)}(t)| : t \in I\}, \quad K_2 = \{\phi^{(3)}(t)/|\phi^{(3)}(t)| : t \in I\}$$

are disjoint closed intervals on the unit circle in R^2 . If $i = 1$ or 2 , define Γ_i to be the sector $\{rv : r \geq 0, v \in K_i\}$. The hypotheses of Theorem 3 imply that

$\Gamma_2 \cap (\Gamma_1 \cup -\Gamma_1) = \{0\}$ and thus that the Γ_i are convex. By (10), for $i = 1$ or 2 the set $\phi^{(i+1)}(I)$ lies in a proper closed subset Γ'_i of $\Gamma_i \setminus \{0\}$, and Γ'_i can be chosen to be convex. Fix $\delta > 0$ such that

(α) $\{x \in R^2: |x| < \delta\} \cap \Gamma'_i = \emptyset$ for $i = 1, 2$;

(β) $b - a \leq \delta^{-1}$;

(γ) For $u \in [a - b, b - a]$ and $x \in R^2$, I_u splits into disjoint subintervals $J_1, \dots, J_K$ with $K \leq \delta^{-1}$ such that $x \cdot (\phi^{(2)}(t+u) - \phi^{(2)}(t))$ is of constant sign on each t -interval J_n . (This is where we use the hypothesis that ϕ_1 and ϕ_2 be both polynomial or trigonometric functions.)

Now fix $u \in [a - b, b - a]$. Define $\Psi(t)$ on I_u by

$$\Psi(t) = \frac{1}{u}(\phi(t+u) - \phi(t)).$$

Letting Γ_1, Γ_2 , and δ be as above, we claim that $\Psi(t)$ satisfies the hypotheses of the lemma. The hypothesis (i) for $\Psi'(t)$ follows from

$$\Psi'(t) = \frac{1}{u} \int_t^{t+u} \phi^{(2)}(s) ds$$

combined with the convexity of Γ'_1 and (α). The hypothesis (i) for $\Psi^{(2)}(t)$ follows similarly. The hypothesis (ii) is a consequence of (γ), and the hypothesis that $\text{length}(J)$ not exceed δ^{-1} is covered by (β). Thus it follows, with the notation of the lemma, that

$$(11) \quad \|\mu * f\|_3 \leq C \|f\|_{3/2}.$$

For functions f on R^2 and $u \in R$, put $D_u f(x) = f(ux)$, $x \in R^2$. Then

$$\lambda_u * f(x) = \int_{I_u} f(x - u\Psi(t)) dt = \int_{I_u} f\left(u\left(\frac{x}{u} - \Psi(t)\right)\right) dt = D_{1/u}(\mu * (D_u f))(x).$$

Now (9) follows from (11) and the fact that

$$\|D_u f\|_p = |u|^{-2/p} \|f\|_p.$$

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