THE ANALYTIC RADON-NIKODYM PROPERTY IN LEBESGUE BOCHNER FUNCTION SPACES

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ABSTRACT. Let X be a complex Banach space, (Ω, Σ, μ) a finite measure space, and $1 \leq p < \infty$. Then $L_p(\mu; X)$ has the analytic Radon-Nikodym property if and only if X has it.

1. Introduction. In [2] Bukhvalov and Danilevich introduced the notion of the analytic Radon-Nikodym property. The relationship between the analytic Radon-Nikodym property and the existence of bounded holomorphic embeddings has been studied by Aurich [1], while its relationship with representable operators has been studied by Dowling [5]. Recently, Edgar [7] has shown that a complex Banach space has the analytic Radon-Nikodym property if and only if every L_1 -bounded analytic martingale in X converges. In this note we are going to show that if a complex Banach space X has the analytic Radon-Nikodym property then so does $L_p(\mu; X)$ for $1 \leq p < \infty$, where (Ω, Σ, μ) is a finite measure space.

2. Preliminaries.

DEFINITION 1. A complex Banach space X has the analytic Radon-Nikodym property if every X -valued measure of bounded variation, defined on the Borel subsets of $\mathbf{T}$ ($= \{z \in \mathbf{C} : |z| = 1\}$), whose negative Fourier coefficients vanish, has a Radon-Nikodym derivative with respect to normalized Lebesgue measure $d\theta/2\pi$ on $\mathbf{T}$.

DEFINITION 2. Let X be a complex Banach space. The space $H^p(X)$ consists of all holomorphic functions $f: D \rightarrow X$, where D is the open unit disc in $\mathbf{C}$, and satisfying $\|f\|_p < \infty$ where

$$\|f\|_p = \sup_{0 \leq r < 1} \left(\int_0^{2\pi} \|f(re^{i\theta})\|^p \frac{d\theta}{2\pi} \right)^{1/p}, \quad 0 < p < \infty,$$

$$\|f\|_\infty = \sup_{z \in D} \|f(z)\|.$$

For details on vector-valued holomorphic functions see [4, 8]. $(H^p(X), \|\cdot\|_p)$ is a Banach space for $1 \leq p \leq \infty$.

THEOREM 1 [2]. *Let X be a complex Banach space. The following are equivalent:*

- (a) X has the analytic Radon-Nikodym property,
- (b) for some p , $1 \leq p \leq \infty$, every function in $H^p(X)$ has radial limits a.e.,
- (c) for all p , $1 \leq p \leq \infty$, every function in $H^p(X)$ has radial limits a.e.,

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(d) for some p , $1 \leq p < \infty$, $\lim_{r \uparrow 1} f_r$ exists in $L_p(\mathbf{T}; X)$ for every $f \in H^p(X)$, where $f_r(e^{i\theta}) = f(re^{i\theta})$,

(e) for all p , $1 \leq p < \infty$, $\lim_{r \uparrow 1} f_r$ exists in $L_p(\mathbf{T}; X)$ for every $f \in H^p(X)$.

3. The analytic Radon-Nikodym property for $L_p(\mu; X)$.

THEOREM 2. Let X be a complex Banach space, $1 \leq p < \infty$, and (Ω, Σ, μ) a finite measure space. Then $L_p(\mu; X)$ has the analytic Radon-Nikodym property if X does.

PROOF. Let $f \in H^p(L_p(\mu; X))$. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n \in L_p(\mu; X)$ for all $n \in \mathbf{N}$. Since f is holomorphic we have that, for each $0 < r < 1$,

$$\sum_{n=0}^{\infty} \|a_n\| r^n < \infty,$$

that is, for each $0 < r < 1$,

$$\sum_{n=0}^{\infty} \left(\int_{\Omega} \|a_n(\omega)\|^p d\mu(\omega) \right)^{1/p} r^n < \infty.$$

Hence, for each $0 < r < 1$,

$$\sum_{n=0}^{\infty} \left(\int_{\Omega} \|a_n(\omega)\|^p d\mu(\omega) \right) r^n < \infty,$$

and so, for each $0 < r < 1$,

$$\int_{\Omega} \left(\sum_{n=0}^{\infty} \|a_n(\omega)\|^p r^n \right) d\mu(\omega) < \infty.$$

Now, for each $0 < r < 1$,

$$\sum_{n=0}^{\infty} \|a_n(\omega)\|^p r^n < \infty,$$

for almost all $\omega \in \Omega$, and from this we get that, for each $0 < r < 1$,

$$\sum_{n=0}^{\infty} \|a_n(\omega)\| r^n < \infty,$$

for almost all $\omega \in \Omega$. Thus, the function $f(\cdot)(\omega): D \rightarrow X$, defined by $f(z)(\omega) = \sum_{n=0}^{\infty} a_n(\omega) z^n$, is holomorphic for almost all $\omega \in \Omega$.

$$\infty > \|f\|_p^p = \sup_{0 \leq r < 1} \int_0^{2\pi} \|f(re^{i\theta})\|^p \frac{d\theta}{2\pi} = \lim_{r \uparrow 1} \int_0^{2\pi} \|f(re^{i\theta})\|^p \frac{d\theta}{2\pi}$$

(since $\|f(\cdot)\|$ is subharmonic)

$$\begin{aligned} &= \lim_{r \uparrow 1} \int_0^{2\pi} \int_{\Omega} \|f(re^{i\theta})(\omega)\|^p d\mu(\omega) \frac{d\theta}{2\pi} \\ &= \lim_{r \uparrow 1} \int_{\Omega} \int_0^{2\pi} \|f(re^{i\theta})(\omega)\|^p \frac{d\theta}{2\pi} d\mu(\omega) \\ &= \int_{\Omega} \left(\lim_{r \uparrow 1} \int_0^{2\pi} \|f(re^{i\theta})(\omega)\|^p \frac{d\theta}{2\pi} \right) d\mu(\omega) \end{aligned}$$

(by the Monotone convergence theorem)

$$\begin{aligned} &= \int_{\Omega} \left(\sup_{0 \leq r < 1} \int_0^{2\pi} \|f(re^{i\theta})(\omega)\|^p \frac{d\theta}{2\pi} \right) d\mu(\omega) \\ &= \int_{\Omega} \|f(\cdot)(\omega)\|_p^p d\mu(\omega). \end{aligned}$$

Therefore, for almost all $\omega \in \Omega$, $\|f(\cdot)(\omega)\|_p < \infty$; that is, for almost all $\omega \in \Omega$, $f(\cdot)(\omega) \in H^p(X)$. By Theorem 1 and the assumption that X has the analytic Radon-Nikodym property we have that, for almost all $\omega \in \Omega$, $\lim_{r \uparrow 1} f_r(\cdot)(\omega)$ exists in $L_p(\mathbf{T}; X)$. Moreover,

$$\begin{aligned} \|f_r - f_s\|_{L_p(\mathbf{T}; L_p(\mu; X))}^p &= \int_0^{2\pi} \|f(re^{i\theta}) - f(se^{i\theta})\|^p \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \int_{\Omega} \|f(re^{i\theta})(\omega) - f(se^{i\theta})(\omega)\|^p d\mu(\omega) \frac{d\theta}{2\pi} \\ &= \int_{\Omega} \int_0^{2\pi} \|f(re^{i\theta})(\omega) - f(se^{i\theta})(\omega)\|^p \frac{d\theta}{2\pi} d\mu(\omega) \\ &= \int_{\Omega} (\|f_r(\cdot)(\omega) - f_s(\cdot)(\omega)\|_{L_p(\mathbf{T}; X)})^p d\mu(\omega). \end{aligned}$$

But

$$\|f_r(\cdot)(\omega) - f_s(\cdot)(\omega)\|_{L_p(\mathbf{T}; X)} \rightarrow 0$$

as $r, s \uparrow 1$ for almost all $\omega \in \Omega$ and all

$$\|f_r(\cdot)(\omega) - f_s(\cdot)(\omega)\|_{L_p(\mathbf{T}; X)} \leq 2\|f(\cdot)(\omega)\|_p.$$

Hence by the dominated convergence theorem,

$$\|f_r - f_s\|_{L_p(\mathbf{T}; L_p(\mu; X))}^p \rightarrow 0$$

as $r, s \uparrow 1$. Thus $\lim_{r \uparrow 1} f_r$ exists in $L_p(\mathbf{T}; L_p(\mu; X))$ and so, by Theorem 1, $L_p(\mu; X)$ has the analytic Radon-Nikodym property.

REMARK. In [3] it is shown that $L_p(\mu; X)$ has the Radon-Nikodym property if X has it and if $1 < p < \infty$. Theorem 2 is the analogue of this result but the method of proof is quite different.

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