EXPONENTIATION OF REALS: EFFECTS OF BASE CHOICE

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ABSTRACT. Let $r > 1$ and X_r be the minimal set of reals containing 1 and closed under $\exp_r$: $x \mapsto r^x$ and addition. The behavior of X_r is studied. In particular among possible order types of X_r there are $\omega, \omega^\omega, \omega + q, \omega + 1 + q$, where q is the dense countable order without endpoints.

Let $r > 0$ and X_r be the minimal set of reals containing 1 and closed under $\exp_r$: $x \mapsto r^x$ and addition. Obviously X_r is dense in $\mathbf{R}_+$ for $r < 1$ and $X_1 = \mathbf{N}$. I would like to describe the behavior of X_r for $r > 1$.

Preliminary remarks. (Proofs are provided in §1.) (i) X_r = the set of values of constant $(1, +, \cdot, \exp_r)$ -terms.

(ii) If X is a well-ordered subset of $\mathbf{R}_+$ closed under addition (or multiplication) then the order type of X is ω^{ω^α} for some α .

(iii) For any countable α if X_r is well-ordered and its order type is $< \omega^{\omega^\alpha}$ then there exists a set $X \subset \mathbf{R}_+$ containing X_r , closed under $+, \cdot, \exp_r$ and of order type ω^{ω^α} .

PROPOSITION 1. Let $a = (a^a + 1)^{1/(a^a + 1)}$ and $b = s^{1/s}$, where s is the root of the equation $x^{1/x} = (x + 1)^{1/(x+1)}$. (One can easily see that both equations have unique roots.) Then

- (i) If $e^{1/e} < r$ then X_r has order type ω .
- (ii) If $a \leq r \leq e^{1/e}$ then X_r and $\overline{X}_r$ have order type ω^ω .
- (iii) If $b < r < a$ then X_r is not well-ordered and the Cantor derivative $(X_r)^{(\omega)} = \emptyset$.
- (iv) X_b is not well-ordered.

The Cantor derivative mentioned in (iii) is defined as follows: $Y^{(0)} = \overline{Y}$, $Y^{(\lambda+1)} =$ the set of limit points of $Y^{(\lambda)}$, and $Y^{(\lambda)} = \bigcap \{Y^{(\mu)} \mid \mu < \lambda\}$ for limit λ .

Some numerical values:

$$e^{1/e} = 1.4446678\dots, \quad a = 1.4446575\dots, \quad b = 1.4360782\dots$$

CONJECTURE. If $1 < r \leq b$ then $\overline{X}_r = \{a \text{ sequence increasing to } g_r\} \cup [g_r, \infty)$, where g_r is the smallest root of the equation $r^x = x$.

PROPOSITION 2. (i) If $1 < r < b$ then X_r contains a perfect subset i.e. a nonempty subset without isolated points.

(ii) The conjecture is valid for $1 < r < 1.4360782\dots$

(iii) If $\ln(\ln(g_b))/\ln(\ln(g_b + 1))$ is irrational then the conjecture is valid for $r = b$.

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Ends of proofs of claims and lemmata are marked with $\square$.

1. Proofs of preliminary remarks. (i) I have to construct for any constant $(1, +, \cdot, \exp_r)$ -term t a constant $(1, +, \exp_r)$ -term u with the same value. Proceed by induction on the number of operations in t . Let $t = t_1 t_2$, where t_1, t_2 are $(1, +, \exp_r)$ -terms and $t_1 \neq 1 \neq t_2$. If e.g. $t_2 = u + v$ then we can apply the induction assumption to $t_1 u$, $t_1 v$ in $(t_1 u) + (t_1 v)$. If both t_1, t_2 are exponents, $t_1 = r^u$, $t_2 = r^v$, then the value of t is the value of $\exp_r(u + v)$. $\square$

(ii) Let τ be the order type of X . It suffices to prove that if $\omega^\lambda < \tau$ then $\omega^{\lambda^2} < \tau$ for any λ , see [2, VII.7]. Let X be closed under addition (the case of X closed under multiplication is quite similar). Let A be a bounded subset of X with order type ω^λ and $x = \sup(A)$; for each $a \in A$ let a^+ be its successor. Then for any $a \in A$ the order type of $\{a^+ + b \mid b \in A \text{ \& } a + x < a^+ + b\}$ is ω^λ , see [2, VII.6] (any nonzero remainder of ω^λ is ω^λ). Thus $A + A$ has order type at least ω^{λ^2} and is bounded by $2x$. $\square$

(iii) I will prove the following statement: for any set $A \subset \mathbf{R}_+$ containing 1 and closed under $+$, $\cdot$, $\exp_r$ with order type ω^{ω^α} , for any $p \in \mathbf{R}_+$, for any countable ordinal β there exists a set $B \subset \mathbf{R}_+$ closed under $+$, $\cdot$, $\exp_r$ with order type $\omega^{\omega^{\alpha+\beta}}$ and satisfying $B \supset A$ & $A \supset B \cap (0, p)$.

Proceed by induction on β . Consider first the limit case. Let $p > 0$ and $\beta = \lim\{\beta_1 + \dots + \beta_n\}_n$ with nonzero β_i 's. Pick up $\{B_i\}_i$ as follows: let $B_0 = A$ and B_{i+1} be a set $\subset \mathbf{R}_+$ such that $B_{i+1} \supset B_i$, $B_{i+1} \cap (0, p + i + 1) \subset B_i$, B_{i+1} is closed under $+$, $\cdot$, $\exp_r$ and its order type is ω^{λ_i} , where $\lambda_i = \omega^{\alpha+\beta_1+\dots+\beta_{i+1}}$. Put $B = \bigcup_i B_i$. Since any bounded subset of B is contained in one of B_i 's B is well-ordered and has order type at most $\omega^{\omega^{\alpha+\beta}}$, and since B contains all B_i 's its order type is at least $\omega^{\omega^{\alpha+\beta}}$.

Now turn to the case of $\beta = 1$. Without loss of generality we can assume that $p > 1$ and $r^x > x + 1$ for $x \geq p$. Let K be a subset of $(p, p + 1)$ with order type ω^{ω^α} and B be the minimal subset of $\mathbf{R}_+$ closed under $+$, $\cdot$, $\exp_r$ and containing $A \cup K$. Since any bounded subset of B is contained in a finite union of $(+, \cdot, \exp_r)$ -combinations of A, K the result follows from [3]:

Let $X, Y \subset \mathbf{R}_+$ be well-ordered; then the order type of $X \cup Y$ is $\leq$ the Hessenberg natural sum of the order types of X and Y , and the order types of $X + Y$ and $X \cdot Y$ are $\leq$ the Hessenberg natural product of the order types of X and Y .

(This result is not very difficult, and you can prove it for your own pleasure.) $\square$

2. Proof of Proposition 1. For $r < e^{1/e}$ let $g_r < h_r$ be the roots of the equation $x^{1/x} = r$ (for $r = e^{1/e}$ there is unique root $x = e$ and for $r > e^{1/e}$ there are no roots). Clearly $g_r < e < h_r$, both g_r and h_r depend on $r \in (1, e^{1/e})$ differentiably and monotonically and tend to e for $r \rightarrow e^{1/e}$, and $r^x > x$ for $x \notin [g_r, h_r]$ and $r^x < x$

for $x \in (g_r, h_r)$. For convenience put also $g_r = h_r = e$ for $r = e^{1/e}$. Denote $\exp_r^0: x \mapsto x$ and $\exp_r^{n+1}(x) = \exp_r(\exp_r^n(x))$. The sequence $\{\exp_r^n(1)\}_n$ increases and if bounded tends to g_r .

Claim 1. Let T be an infinite set of constant $(1, g_r, +, \exp_r)$ -terms. If the set of values of terms $\in T$ is bounded then T contains terms with arbitrarily high $\exp_r$ -nests.

PROOF. If values of terms $\in T$ are bounded then there exists an n such that each $t \in T$ consists of at most n summands of the forms $1, g_r, r^u$. Thus the set of such summands is still infinite and the values of u 's are bounded. $\square$

This provides (i) of the proposition: if t contains an $\exp_r$ -nest of height n then $t \geq \exp_r^n(1)$, and $\{\exp_r^m(1)\}_m$ is unbounded for $r > e^{1/e}$. On the other hand if $r \leq e^{1/e}$ then g_r is a limit point for X_r and since X_r is closed under addition all its finite number Cantor derivatives are nonempty.

Claim 2. Let $b < r \leq e^{1/e}$ and $\{t_n\}_n$ be a sequence of constant $(1, g_r, +, \exp_r)$ -terms whose values are bounded. Then there exist finite sets U of constant $(1, +, \exp_r)$ -terms and V of terms in variables and $1, g_r, +, \exp_r$ such that each $t_n = v(w_1, \dots, w_l)$ for some $v \in V$, where each $w_i = \exp_r^m(u)$ for some $u \in U \cup \{g_r\}$ and m .

PROOF. Note first that there is a k such that $\exp_r^k(1) + 1 > h_r$ (this follows from $r > b$: $h_b = 1 + g_b = 1 + \lim\{\exp_b^m(1)\}_m$). Let K be the least such k . Note also that $\{\exp_r^m(x)\}_m$ is unbounded for $x > h_r$. Let the values of t_n 's be $< M$ and J be the least j such that $\exp_r^j(\exp_r^K(1) + 1) > M$.

Let $U = \{1\} \cup \{u | u \text{ is a sum and appears as a subterm of some } t_n \text{ in the scope of at least } J \exp_r \text{'s}\}$. I will show that U is finite. Note that the values of terms $\in U$ are bounded by $\exp_r^K(1) + 1$. If U was infinite then Claim 1 would tell us that elements of U contain arbitrarily high $\exp_r$ -nests which is impossible in view of the above bound and the fact that the nontrivial elements of U are sums. If a term $u \in U$ contained g_r we would have $u = q + s \geq g_r + 1 > \exp_r^K(1) + 1$; thus elements of U are $(1, +, \exp_r)$ -terms.

To obtain V imagine t_n 's typewritten on sheets of paper so that each $\exp_r(w)$ is displayed as r^w and substitute by variables the parts of t_n 's which are above J . One can express this rigorously as follows. For a subterm u of w let $\text{height}(u, w) =$ the number of $\exp_r$'s in w affecting u ; and let $\text{height}(w) = 1 + \max\{\text{height}(u, w) | u \text{ is a subterm of } w\}$. Present each t_n with height J in the form $v(w_1, \dots, w_l)$, where v is a term in variables and $1, g_r, +, \exp_r$, w_i 's are constant $(1, g_r, +, \exp_r)$ -terms, and $\text{height}(v) = J + 1$, $\text{height}(\text{variable}_i, v) = \text{height}(w_i, t_n) = J$ for each i . Let V be the set of such v 's. To show that V is finite note first that the values of $\{v(1, \dots, 1) | v \in V\}$ are bounded ($< M$); now reasoning similar to that in the proof of Claim 1 provides the result. $\square$

COROLLARY 3. If $b < r \leq e^{1/e}$ then $(X_r)^{(m)} = \{\text{the values of constant } (1, g_r, +, \exp_r)\text{-terms with } \geq m \text{ occurrences of } g_r\}$ for any m , and $(X_r)^{(\omega)} = \emptyset$.

PROOF. Proceed by induction on n . Let $x \in (X_r)^{(n+1)}$,

$$x = \lim\{\text{value of } t_m\}_m,$$

each t_m being a constant $(1, g_r, +, \exp_r)$ -term with $\geq n$ occurrences of g_r and

the values of t_m 's distinct. In view of Claim 2 we can assume that each $t_m = u(v_m, \dots, w_m)$ for a term u in variables and $1, g_r, +, \exp_r$ with $\geq n$ occurrences of g_r and $v_m = \exp_r^{j(m)}(v), \dots, w_m = \exp_r^{k(m)}(w)$ for some constant $(1, +, \exp_r)$ -terms $v, \dots, w$ with increasing $j(\cdot), \dots, k(\cdot)$. Thus x = the value of $u(g_r, \dots, g_r)$. Conversely the value of $t(g_r)$ is $\lim\{\text{the value of } t(\exp_r^m(1))\}_m$. Besides, Claim 2 shows that if $\{t_m\}_m$ is a bounded sequence of constant $(1, g_r, +, \exp_r)$ -terms then $\{\text{the number of occurrences of } g_r \text{ in } t_m\}_m$ is bounded, whence $(X_r)^{(\omega)} = \emptyset$. $\square$

COROLLARY 4. *If $b < r \leq e^{1/e}$ and X_r is well-ordered then its and $\overline{X}_r$'s order types are both ω^ω .*

PROOF. Immediately from the preliminary remark (ii) and the above corollary. $\square$

Clearly $r^{r^r+1} < r^r + 1$ for $r < a$ whence $\{\exp_r^m(r^r + 1)\}_m$ decreases for $r < a$ which gives (iii), (iv) of the proposition. I will show that X_r is well-ordered for $r \geq a$.

Claim 5. If X_r is not well-ordered then there exists a sequence $\{t_m\}_m$ of $(1, +, \exp_r)$ -terms such that $t_m > \exp_r(t_{m+1})$ for each m .

PROOF. Let $\{s_m\}_m$ be a decreasing sequence of $(1, +, \exp_r)$ -terms. Put $t_0 = s_0$. Since the values of $\{s_m\}_m$ are bounded $\{\text{the number of summands in } s_m\}_m$ is also bounded. Consequently (turning if necessary to a subsequence) we may assume that there exists a sequence $\{s'_m\}_{m \geq 1}$ of $(1, +, \exp_r)$ -terms such that each $\exp_r(s'_m)$ is a summand of s_m and the values of $\{s'_m\}_{m \geq 1}$ decrease. Of course the value of t_0 is greater than those of $\{\exp_r(s'_m)\}_{m \geq 1}$. Apply the same procedure to $\{s'_m\}_{m \geq 1}$ and proceed ω times. $\square$

Claim 6. Let $b < r \leq e^{1/e}$. If X_r is not well-ordered then there exists a $(1, +, \exp_r)$ -term t such that $r^t < t$.

PROOF. Let $\{t_m\}_m$ be a sequence of $(1, +, \exp_r)$ -terms such that $t_m > \exp_r(t_{m+1})$ for each m (see Claim 5). If $t_m > h_r$ then $\&_k t_{m+k} \leq x_k$, where $x_{k+1} = \log_r(x_k)$, x_0 = the value of t_m ; and $\{x_k\}_k$ converges to h_r . Thus the values of t_m 's are bounded. Since for $k > 0$ inequality $\exp_r^k(x) < x$ holds iff $x \in (g_r, h_r)$ it will be enough to find a t with $t > \exp_r^k(t)$ for some $k > 0$: then $g_r < t < h_r$ and $r^t < t$.

If for some $m < n$ we have $t_m = t_n$ then we have $t_m > \exp_r^{n-m}(t_m)$. Suppose that all t_m 's are distinct. Then by Claim 2 there exist an increasing sequence s of positive integers, a term v in variables and $1, +, \exp_r$, constant $(1, +, \exp_r)$ -terms $u_1, \dots, u_l$ and increasing sequences $k_1, \dots, k_l$ of positive integers such that for each m we have $t_{s(m)} = v(\exp_r^{k_1(m)}(u_1), \dots, \exp_r^{k_l(m)}(u_l))$. If $\exp_r^{k_i(1)}(u_i) > \exp_r^{k_i(2)}(u_i)$ for at least one i then we have $u_i > \exp_r^{k_i(2)-k_i(1)}(u_i)$. Otherwise we have

$$\begin{aligned} t_{s(1)} &> \exp_r^{s(2)-s(1)}(t_{s(2)}) \\ &= \exp_r^{s(2)-s(1)}(v(\exp_r^{k_1(2)}(u_1), \dots, \exp_r^{k_l(2)}(u_l))) \\ &\geq \exp_r^{s(2)-s(1)}(v(\exp_r^{k_1(1)}(u_1), \dots, \exp_r^{k_l(1)}(u_l))) \\ &= \exp_r^{s(2)-s(1)}(t_{s(1)}). \quad \square \end{aligned}$$

Claim 7. If $a \leq r \leq e^{1/e}$ then X_r is well-ordered.

PROOF. Suppose the contrary; let t be a $(1, +, \exp_r)$ -term such that $r^t < t$. Without loss of generality we can assume that t is not an exponent (if $t = r^u$ then

$r^{r^u} < r^u$ implies $r^u < u$). Consequently t is a sum. If t has three or more summands then its value is $\geq 3 > h_a \geq h_r$ contradicting the choice of t (for $x > h_r$ implies $r^x > x$). Thus t has two summands. If both summands of t are $\neq 1$ then they are $\geq r$ whence $t \geq 2r \geq 2.889 \dots > h_a \geq h_r$ (again impossible). On the other hand $2 < g_a \leq g_r$ whereas $r^x > x$ for $x < g_r$. Thus $t = 1 + r^u$. If u is a sum then $t \geq 1 + r^2 > 1 + 2 > h_a \geq h_r$. If $u = 1$ then the value of t is $1 + r \leq 2.444 \dots < g_a \leq g_r$. Thus $u = r^v$. If v is a sum then $t \geq 1 + r^{r^2} > 1 + r^2 > 3 > h_a \geq h_r$. If $v = r^w$ then $t \geq 1 + r^{r^r} \geq 1 + a^{a^a} = 2.8699218 \dots > h_a \geq h_r$. Thus $v = 1$ and $t = 1 + r^r$. However $\exp_r(1 + r^r) \geq 1 + r^r$ for $r \geq a$ by definition of a . $\square$

3. Proof of Proposition 2.

LEMMA 1. Let $f \in C^1([p, q])$, $f(p) = p$, $f(q) \geq p + 1$ and $0 < f'(x) < 1$ for $x \in (p, q)$. Let Y be the subset of $[p, q]$ minimal with respect to the properties: $p \in Y$, $x \in Y \& x + 1 < q \Rightarrow x + 1 \in Y$, $x \in Y \Rightarrow f(x) \in Y$. Then $\overline{Y} = [p, q]$.

PROOF. Suppose the contrary. Any open interval $\subset (p, q) - \overline{Y}$ will be called a hole. Since $\overline{Y}$ is closed under $+1, f$ we have $\overline{Y} \supset [p, q] \cap (\bigcup_{m=0}^{\infty} m + f(\overline{Y}))$ whence the length of any hole is bounded by that of a hole $\subset (p, p + 1) \subset (p, f(q))$. On the other hand f maps (p, q) onto $(p, f(q))$ monotonically with derivative < 1 , so any hole $\subset (p, f(q))$ is the f -image of a larger one. Thus we obtain a contradiction: let H be a hole with maximal length, let H' be a hole $\subset (p, p + 1)$ with $\text{length}(H') \geq \text{length}(H)$, let $H'' = f^{-1}(H')$, then $\text{length}(H'') > \text{length}(H)$. $\square$

COROLLARY 2. Let $c = s^{1/s}$, where s is the root of the equation $x + 1 = x/\ln(x)$. Then $[g_r, \infty) \subset \overline{X}_r$ for $1 < r < c$ ($c = 1.3998168 \dots$).

PROOF. Put $d = -\ln(\ln(r))/\ln(r)$. Then $(r^x)' = r^x \cdot \ln(r) < 1$ for $x < d$ and $g_r + 1 \leq r^d < d < h_r$ for $r < c$. So the result follows from Lemma 1 applied to $\exp_r \in C^1([g_r, d])$. $\square$

Now let us turn to $r \in (c, b)$. A real $x \in [g_r + 1, \log_r(g_r + 1))$ will be called tame if it is the value of a constant $(1, g_r, +, \exp_r)$ -term. Clearly $\{\text{tame reals}\} \subset \overline{X}_r$.

Claim 3. If the set of tame reals is dense in $(g_r + 1, \log_r(g_r + 1))$ then X_r is dense in (g_r, ∞) .

PROOF. If tame reals are dense in $(g_r + 1, \log_r(g_r + 1))$ then X_r is also dense there and consequently X_r is dense in $\bigcup_{k=0}^{\infty} \exp_r^k([g_r + 1, \log_r(g_r + 1)])$ which is just $(g_r, \log_r(g_r + 1))$ since $g_r = \lim\{\exp_r^k(g_r + 1)\}_k$ for $r < b$. Thus X_r is dense in $[g_r, g_r + 1) \subset [g_r, \log_r(g_r + 1))$ and so X_r is dense in $[g_r, \infty)$. $\square$

So, I intend to prove that tame reals are dense in $[g_r + 1, \log_r(g_r + 1))$ for $c \leq r < 1.4360782 \dots$ and I conjecture that this will hold for any $r \in (c, b)$. For any sequence $m = \{m_k\}_{k \geq 1}$ of positive integers let $\{m\mathcal{L}_k\}_k$ be the following sequence of one-variable $(1, +, \exp_r)$ -terms: $m\mathcal{L}_0(x) = x$, $m\mathcal{L}_k(x) = m\mathcal{L}_{k-1}(\exp_r^{m_k}(x) + 1)$ for $k \geq 1$.

Claim 4. For any $x \in (g_r + 1, \log_r(g_r + 1))$ there exist unique $l \in \mathbb{N}$ and $y \in [g_r + 1, \log_r(g_r + 1))$ such that $x = \exp_r^l(y) + 1$. For any nontame $x \in [g_r + 1, \log_r(g_r + 1))$ there exist unique sequences $\{m_k\}_{k \geq 1}$ of positive integers and $\{y_k\}_{k \geq 0}$ of reals $\in (g_r + 1, \log_r(g_r + 1))$ such that $x = m\mathcal{L}_k(y_k)$ for each k .

PROOF. The first part of the claim is an immediate consequence of the facts that $\log_r(g_r + 1) - (g_r + 1) < 1$ for $r \geq c$ (existence) and that $(g_r, \log_r(g_r + 1))$ is the disjoint union of $\{\exp_r^l[g_r + 1, \log_r(g_r + 1))\}_{l=0}^{\infty}$ for $r < b$ (uniqueness). For the

second part of the claim proceed by induction: if we have already y_k (put $y_0 = x$) then $y_k \neq g_r + 1$ (since x is not tame) and by the first part of the claim we get m_{k+1} and y_{k+1} . $\square$

Claim 5. Tame reals are dense in $(g_r + 1, \log_r(g_r + 1))$ for $c \leq r < 1.4360782 \dots$ (this completes (ii) of Proposition 2).

PROOF. Let $x \in [g_r + 1, \log_r(g_r + 1))$ be nontame and $\{m_k\}_k, \{y_k\}_k$ be as in Claim 4. We have $x = m\mathcal{L}_k(y_k)$ for each k and I am going to prove that $x = \lim\{m\mathcal{L}_k(g_r + 1)\}_k$. Note that functions $\{m\mathcal{L}_k\}_k$ and their derivatives are positive and increasing (except the trivial case $k = 0$). Since

$$\begin{aligned} m\mathcal{L}_k(y_k) - m\mathcal{L}_k(g_r + 1) &< (y_k - g_r - 1) \cdot \left(\frac{d}{dt} m\mathcal{L}_k(t) \Big|_{t=y_k} \right) \\ &< (\log_r(g_r + 1) - g_r - 1) \cdot \left(\frac{d}{dt} m\mathcal{L}_k(t) \Big|_{t=y_k} \right) \end{aligned}$$

and $\log_r(g_r + 1) - g_r - 1 < 1$ for $r \geq c$, it suffices to show that $(m\mathcal{L}_k(t))' \Big|_{t=y_k}$ tends to 0 while $k \rightarrow \infty$. Since

$$\begin{aligned} (m\mathcal{L}_k(t))' \Big|_{t=y_k} &= (m\mathcal{L}_{k-1}(\exp_r^{m_k}(t) + 1))' \Big|_{t=y_k} \\ &= (m\mathcal{L}_{k-1}(t))' \Big|_{t=y_{k-1}} \cdot (\exp_r^{m_k}(t) + 1)' \Big|_{t=y_k} \end{aligned}$$

and

$$(\exp_r^{m_k}(t) + 1)' \Big|_{t=y_k} < (\exp_r^{m_k}(t) + 1)' \Big|_{t=\log_r(g_r + 1)}$$

it will be enough to prove that $\sup\{(\exp_r^{m_k}(t))' \Big|_{t=\log_r(g_r + 1)}\}_k < 1$. Well,

$$(\exp_r^n(t))' = \prod_{j=1}^n (\ln(r) \cdot \exp_r^j(t))$$

and its factors $\{\ln(r) \cdot \exp_r^j(t)\}_j$ decrease with increasing j for any fixed $t \in (g_r, h_r)$. Consequently if $(\exp_r^n(t))' < 1$ at some $t \in (g_r, h_r)$ then $(\exp_r^m(t))' < 1$ at the same t whenever $m \geq n$. Let $J = J(r)$ be the minimal positive integer with the property $\exp_r^J(g_r + 1) + 1 < \log_r(g_r + 1)$. Then $m_k \geq J$ for all k and the claim will go through whenever $(\exp_r^J(t))' \Big|_{t=\log_r(g_r + 1)} < 1$. I checked this by machine computations (with long reals) for $c \leq r < 1.4360782 \dots$ (corresponding to $J \leq 150$). $\square$

REMARK. Actually I conjecture that the inequality $(\exp_r^{J(r)}(t))' \Big|_{t=\log_r(g_r + 1)} < 1$ (see above) holds for all $r \in (c, b)$. Moreover, it is possible to extend $\{\exp_r^k\}_k$ to a one-parameter iteration subgroup of the diffeomorphisms group of (g_r, h_r) , and I conjecture that

$$\frac{d}{dr} \left(\frac{d}{dt} \exp_r^{j(r)}(t) \text{ at } t = \log_r(g_r + 1) \right) < 0 \quad \text{for } c \leq r < b,$$

where $j(r)$ is a differentiable function of r determined by equation $\exp_r^j(g_r + 1) + 1 = \log_r(g_r + 1)$.

Claim 6. There exists $K \in \mathbb{N}$ such that $(\exp_r^M(t))' \Big|_{t=\log_r(g_r + 1)} < 1$ for $M \geq K$.

PROOF. $(\exp_r^M(t))' = \prod_{j=1}^M (\ln(r) \cdot \exp_r^j(t))$ and $\lim\{\ln(r) \cdot \exp_r^{j-1}(g_r + 1)\}_j = \ln(r) \cdot g_r = \ln(g_r) < 1$. Consequently $\ln(r) \cdot \exp_r^{j-1}(g_r + 1) < \text{const} < 1$ for sufficiently large j 's and $\{(d/dt) \exp_r^M(t) \Big|_{t=\log_r(g_r + 1)}\}_M$ tends to 0 for $M \rightarrow \infty$. $\square$

Claim 7. For any $r \in (c, b)$ there is an embedding of the Cantor binary set $\{0, 1\}^{\mathbb{N}}$ into $\overline{X}_r$. (This provides (i) of the proposition.)

PROOF. Take $K \in \mathbf{N}$ so large that $(\exp_r^K(t))'|_{t=\log_r(g_r+1)} < 1$ (see Claim 6) and $\exp_r^{K-1}(g_r+1)+1 < \log_r(g_r+1)$. The latter implies that for any sequence $\{m_k\}_k$ of positive integers satisfying $\&_k m_k \geq K$ we have

$$\&_k m\mathcal{L}_k[g_r+1, \log_r(g_r+1)) \subseteq [g_r+1, \log_r(g_r+1)).$$

Evidently $\{m\mathcal{L}_k(g_r+1)\}_k$ is increasing and in view of the above inclusion is also bounded. For any 0-1-sequence $\mathbf{l} = \{l_k\}_{k=1}^\infty$ let $f(\mathbf{l}) = \lim\{m\mathcal{L}_k(g_r+1)\}_k$, where $m_k = K + l_k$. It remains to check that f is continuous and injective.

Injectivity. Recall that $f(\{0,1\}^{\mathbf{N}}) \subset (g_r+1, \log_r(g_r+1))$. Let $\mathbf{k}, \mathbf{l} \in \{0,1\}^{\mathbf{N}}$ coincide in the first $n-1$ entries and $k_n = 0, l_n = 1$. Let $\mathbf{k}_q, \mathbf{l}_q$ denote the $(q+1)$ -tails of $\mathbf{k}, \mathbf{l}$ respectively i.e. $(\mathbf{k}_q)_p = k_{q+p}, (\mathbf{l}_q)_p = l_{q+p}$ for $q \in \mathbf{N}$. We have $f(\mathbf{k}_n), f(\mathbf{l}_n) \in (g_r+1, \log_r(g_r+1))$ and

$$\begin{cases} f(\mathbf{k}_{n-1}) \in 1 + \exp_r^K(g_r+1, \log_r(g_r+1)), \\ f(\mathbf{l}_{n-1}) \in 1 + \exp_r^{K+1}(g_r+1, \log_r(g_r+1)) \end{cases}$$

with disjoint right-hand side sets. Since $p\mathcal{L}_{n-1}$, where p is $K+(\text{the common beginning of } \mathbf{k}, \mathbf{l})$, is a strictly increasing function and $f(\mathbf{k}) = p\mathcal{L}_{n-1}(f(\mathbf{k}_{n-1}))$, $f(\mathbf{l}) = p\mathcal{L}_{n-1}(f(\mathbf{l}_{n-1}))$ we have $f(\mathbf{k}) \neq f(\mathbf{l})$.

Continuity. Denote $(\exp_r^K(t))'|_{t=\log_r(g_r+1)} = w, w < 1$. Then

$$\sup\{(d/dt)m\mathcal{L}_k(t)|t \in (g_r+1, \log_r(g_r+1))\} \leq w^k$$

whenever $\&_k m_k \geq K$. Consequently if $\mathbf{k}, \mathbf{l} \in \{0,1\}^{\mathbf{N}}$ coincide in the first n entries then $|f(\mathbf{k}) - f(\mathbf{l})| < (\log_r(g_r+1) - g_r - 1) \cdot w^n$. $\square$

LEMMA 8. Let $f \in C^2(\mathbf{R})$, $f(0) = 0, f'(0) \in \mathbf{R}_+ - \{1\}$. For any $x_0 \in \mathbf{R}_+$ set $x_k = f(x_{k-1}), k \geq 1$. Then $\forall s > 0: \exists q > 0:$ for any n ,

$$(i) \ x_0(1-s) < x_n(f'(0))^{-n} < x_0(1+s),$$

$$(ii) \ (x_0 - y_0)(1-s) < (x_n - y_n)(f'(0))^{-n} < (x_0 - y_0)(1+s),$$

whenever $0 < y_0 < x_0 < q$ for $f'(0) < 1$ or $0 < y_n < x_n < q$ for $f'(0) > 1$.

PROOF. It suffices to consider only $f'(0) < 1$. We have $x_k = f'(0)x_{k-1} + u_k x_{k-1}^2$ with $u_k = f''(a \text{ real} \in (0, x_{k-1}))/2$ and $\{u_k\}_k$ is bounded. So

$$x_k/x_{k-1} = f'(0) \cdot (1 + u_k x_{k-1}/f'(0))$$

and

$$x_n/x_0 = (f'(0))^n \cdot \prod_{k=1}^n (1 + u_k x_{k-1}/f'(0)).$$

We also have $x_k - y_k = (x_{k-1} - y_{k-1})v_k$ with $v_k = f'(a \text{ real} \in (y_{k-1}, x_{k-1})) = f'(0) + w_k x_{k-1}$ with $w_k = (a \text{ real} \in (0, 1)) \cdot f''(a \text{ real} \in (0, x_{k-1}))$ and $\{w_k\}_k$ is bounded. So

$$(x_n - y_n)/(x_0 - y_0) = (f'(0))^n \cdot \prod_{k=1}^n (1 + w_k x_{k-1}/f'(0)).$$

And since for sufficiently small x_0 we have $\&_n x_n < x_0 \cdot t^n$ for any $t > f'(0)$, both products $\prod_k (1 + u_k x_{k-1}/f'(0))$ and $\prod_k (1 + w_k x_{k-1}/f'(0))$ can be made arbitrarily close to 1 by appropriate choice of q . $\square$

From now on assume that $\ln(\ln(g_b))/\ln(\ln(h_b))$ is irrational. Consider the set

$$W = \{\exp_b^m(\exp_b^n(1) + 1) \mid m, n \in \mathbf{N} \text{ \& } \exp_b^n(1) + 1 > g_b\}.$$

Clearly $W \subset (g_b, h_b)$. I will prove that W is dense in (g_b, h_b) .

Claim 9. For any $s > 0$ there exists a sequence $\{w_k\}_{k \in \mathbf{N}} \subset W$ such that $w_k < w_{k+1}$ & $(h_b - w_b)/(h_b - w_{k+1}) < (1+s)^5/(1-s)^5$ for all k , and $\lim\{w_k\}_k = h_b$.

PROOF. $(b^t)' = b^t \cdot \ln(b)$ which is $= \ln(g_b) < 1$ at $t = g_b$ and $= \ln(h_b) > 1$ at $t = h_b$. Let $s < 1$. Apply Lemma 8 to b^t on $(0, g_b]$ (regarded as $g_b - t \mapsto g_b - b^t$) to obtain q_1 and apply Lemma 8 to b^t on $(g_b, h_b]$ (regarded as $h_b - t \mapsto h_b - b^t$) to obtain q_2 ; let $q = \min(q_1, q_2)$. Take the minimal l satisfying $\exp_b^l(1) + 1 > g_b - q \cdot (1-s)^2$. Then for any n we have

$$(1-s)(\ln(g_b))^n < \frac{g_b - \exp_b^{l+n}(1)}{g_b - \exp_b^l(1)} < (1+s)(\ln(g_b))^n$$

and since $h_b = g_b + 1$ for any m, n satisfying $\exp_b^m(\exp_b^{l+n}(1) + 1) > h_b - q$ we have

$$(1-s)(\ln(h_b))^m < \frac{h_b - \exp_b^m(\exp_b^{l+n}(1) + 1)}{g_b - \exp_b^{l+n}(1)} < (1+s)(\ln(h_b))^m;$$

bringing together these inequalities we obtain

$$\begin{aligned} (1-s)^2(\ln(h_b))^m(\ln(g_b))^n &< \frac{h_b - \exp_b^m(\exp_b^{l+n}(1) + 1)}{g_b - \exp_b^l(1)} \\ &< (1+s)^2(\ln(h_b))^m(\ln(g_b))^n. \end{aligned}$$

Now, by the irrationality of $\ln(\ln(g_b))/\ln(\ln(h_b))$ one can choose m, n so as to make $(\ln(h_b))^m(\ln(g_b))^n$ arbitrarily close to any positive real needed, in particular one can choose (m_k, n_k) 's to approximate well from above each real of the form $(1-s)^{3k}/(1+s)^{3k}$, $k \in \mathbf{N}$. The corresponding $w_k = \exp_b^{m_k}(\exp_b^{n_k}(1) + 1)$'s satisfy the requirements of the claim: $(1+s)^{1-\varepsilon}/(1-s)^{1-\varepsilon} < \tilde{w}_k/\tilde{w}_{k+1} < (1+s)^5/(1-s)^5$, where $\tilde{w}_i = h_b - w_i$. (Divide the inequalities for w_i 's.) $\square$

Claim 10. There exists $q > 0$ such that W is dense in $(h_b - q, h_b)$.

PROOF. For $s = 1/2$ apply Lemma 8 to b^t on $(0, g_b]$ to obtain $q_1, q_1 < 1$, and apply Lemma 8 to b^t on $(g_b, h_b]$ to obtain $q_2, q_2 < 1$ (as in the beginning of Claim 9). Put $q = \frac{1}{2} \min(q_1, q_2)$. I will show that for each $u > 0$ there exists an increasing sequence $\{v_k\}_k \subset W$ such that $v_1 < h_b - q$ and $\&_k v_{k+1} - v_k < u$ with $\lim\{v_k\}_k = h_b$.

Apply Claim 9 with such an s that $(w_{k+1} - w_k)/(h_b - w_k) < u/6q$ for each k . Let n be the minimal positive integer such that $\exp_b^n(w_1) < h_b - q$. Then $\{v_k\}_k$, $v_k = \exp_b^n(w_k)$, satisfy the above requirements. $\square$

Since $(g_b, g_b + 1) = (g_b, h_b) = \bigcup_{m=1}^{\infty} \exp_b^m(h_b - q, h_b)$ for any $q \in (0, 1)$ part (iii) of the proposition is completed.

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