ON THE CARDINALITY OF A COMPACT T_1 SPACE

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ABSTRACT. In this paper two theorems on the cardinality of a compact T_1 space are stated. They generalize the following result: an uncountable first countable compact T_1 space has cardinality greater than or equal to $2^{\aleph_0}$.

Malyhin, in 1978 [3], proved that an uncountable first countable compact T_1 space has cardinality greater than or equal to $2^{\aleph_0}$.

In this brief note we give two generalizations of this theorem and ask some questions.

For notation and terminology we follow [1, 2]; in particular, $|X|$ denotes the cardinality of X and, for a given cardinal number m , the successor cardinal is indicated by m^+ .

LEMMA 1. *If X is an infinite compact T_1 space, such that $\psi(X) < |X|$, then for every closed subset $C \subseteq X$ with $|C| \geq \psi(X)^+$ there exist two disjoint closed sets $C_1, C_2 \subseteq C$ such that $|C_1| \geq \psi(X)^+$ and $|C_2| \geq \psi(X)^+$.*

PROOF. Let $m = \psi(X)^+$ and suppose there exists a closed set $C \subseteq X$ with $|C| \geq m$ such that every two closed set $C_1, C_2 \subseteq C$ with $|C_1| \geq m$ and $|C_2| \geq m$ intersect.

Let $\mathcal{F}$ be the family of all closed subsets F of C such that $|F| \geq m$. First we prove that the family $\mathcal{F}$ is closed under finite intersections. To see this, it suffices to show that if $F_1, F_2 \in \mathcal{F}$, then $|F_1 \cap F_2| \geq m$. By contradiction, assume that $|F_1 \cap F_2| < m$. Since $F_1 \cap F_2$ is compact and $|F_1 \cap F_2| \leq \psi(X)$, it follows that $\psi(F_1 \cap F_2, X) \leq \psi(X)$ (see Lemma b, p. 30 in [2]).

Let $\mathcal{U}$ be a family of open sets such that $|\mathcal{U}| = \psi(F_1 \cap F_2, X)$ and $F_1 \cap F_2 = \bigcap_{U \in \mathcal{U}} U$. We have $F_1 = (F_1 \cap F_2) \cup (\bigcup_{U \in \mathcal{U}} (F_1 - U))$. By regularity of m , there exists $U_1 \in \mathcal{U}$ such that $|F_1 - U_1| \geq m$, and in a similar way, there exists $U_2 \in \mathcal{U}$ such that $|F_2 - U_2| \geq m$. Now letting $U = U_1 \cap U_2$, we obtain two disjoint closed subsets of C , $F_1 - U$ and $F_2 - U$, whose cardinality is greater than or equal to m , contradicting our assumption. Since the family $\mathcal{F}$ is closed under finite intersection it has the finite intersection property.

Now, if we fix a point $x \in C$, because $\psi(x, X) < m$, we can conclude, as above, that there exists $F \in \mathcal{F}$, such that $x \notin F$. This implies that $\mathcal{F}$ has an empty intersection, contradicting the compactness of X . Thus the set C cannot exist.

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THEOREM 1. *If X is an infinite compact T_1 space such that $\psi(X) < |X|$, then $|X| \geq 2^{\aleph_0}$.*

PROOF. By Lemma 1, there exist two disjoint closed sets $C_0, C_1 \subseteq X$ such that $|C_0| \geq \psi(X)^+$ and $|C_1| \geq \psi(X)^+$. By a similar argument, we can construct closed sets $C_{00}, C_{01} \subseteq C_0$ and $C_{10}, C_{11} \subseteq C_1$ with the same property, and so on. For every sequence of 0 and 1, $\alpha = \varepsilon_0 \varepsilon_1 \cdots \varepsilon_n \cdots$, put $C_\alpha = C_{\varepsilon_0} \cap C_{\varepsilon_0 \varepsilon_1} \cap \cdots \cap C_{\varepsilon_0 \varepsilon_1 \cdots \varepsilon_n} \cap \cdots$. The compactness of X implies $C_\alpha \neq \emptyset$. Moreover, if $\alpha \neq \beta$, $C_\alpha \cap C_\beta = \emptyset$. For each sequence α choose $x_\alpha \in C_\alpha$ to obtain a 1-1 mapping from $2^{\aleph_0}$ into X and this proves $|X| \geq 2^{\aleph_0}$.

LEMMA 2. *If X is an infinite compact T_1 space, such that for every $x \in X$, $\psi(x, X) < |X|$ and $|X|$ is regular, then for every closed set $C \subseteq X$ with $|C| = |X|$ there exist two disjoint closed sets $C_1, C_2 \subseteq C$ such that $|C_1| = |C_2| = |X|$.*

PROOF. As in the proof of Lemma 1, we assume the existence of a closed subset C of X , with $|C| = |X|$, such that any two closed sets $C_1, C_2 \subseteq C$ must intersect whenever their cardinalities are equal to $|X|$. Let $\mathcal{F}$ be the family of all closed subsets F of C satisfying $|F| = |X|$. As in Lemma 1 we now show that $\mathcal{F}$ has the finite intersection property. To see this, let $F_1, F_2 \in \mathcal{F}$ and assume that $|F_1 \cap F_2| < |X|$. Since for every $x \in F_1 \cap F_2$, $\psi(x, X) < |X|$, $F_1 \cap F_2$ is compact and $|X|$ is regular, it follows that $\psi(F_1 \cap F_2, X) < |X|$. The finite intersection property now follows, exhibiting as in Lemma 1 two disjoint closed subsets of C with the same cardinality as X . From now on the proof of Lemma 2 follows closely the lines of the proof of Lemma 1.

THEOREM 2. *If X is an infinite compact T_1 space such that, for every $x \in X$, $\psi(x, X) < |X|$ and $|X|$ is regular, then $|X| \geq 2^{\aleph_0}$.*

PROOF. Using Lemma 2, the proof is similar to that of Theorem 1.

COROLLARY. *If X is an infinite compact T_1 space such that, for every $x \in X$, $\psi(x, X) < \text{cf}(|X|)$, then $|X| \geq 2^{\aleph_0}$.*

PROOF. If $\text{cf}(|X|) < |X|$ the corollary follows from Theorem 1, and if $\text{cf}(|X|) = |X|$ from Theorem 2.

REMARK. We note that the proofs of Lemmas 1 and 2 become quite trivial in case X is also T_2 . To prove Lemma 1, for example, note first that every infinite compact space C contains at least one point p such that $|C| = |U|$ for every neighborhood U of p . If the space C in question contains at least two such points, choose two and use a pair of disjoint closed neighborhoods for C_1, C_2 . If C has just one such point p , write $\{p\} = \bigcap U \in \mathcal{U}$ with $\mathcal{U}$ a family of open subsets of C such that

$$|\mathcal{U}| = \psi(p, C) \leq \psi(p, X) \leq \psi(X) \leq |C|$$

to deduce that $|C - U| > \psi(X)$ for some $U \in \mathcal{U}$: then for C_1, C_2 take $C - U$ and some closed neighborhood of p contained in U .

In Theorem 2, the assumption that $|X|$ is regular cannot be omitted. To see this, assume that $\aleph_\Omega < 2^{\aleph_0}$ and let τ be the first ordinal number of cardinality $\aleph_\Omega$. The space $[0, \tau]$, with the ordinal topology, is the required example. Analogously, in

Theorem 1, the assumption about the pseudo-character cannot be omitted, even if the cardinality of the space is regular. To see this, it suffices to consider the one point compactification of the discrete space $\mathscr{D}(\aleph_1)$, when we assume $\aleph_1 < 2^{\aleph_0}$.

The two theorems above suggest the following

QUESTION 1. Is it true that if X is an infinite compact T_1 space such that $\psi(X) < |X|$, then $|X| \geq 2^{\log \psi(X)}$?

QUESTION 2. Is it true that if X is an infinite compact T_1 space such that, for every $x \in X$, $\psi(x, X) < |X|$ and $|X|$ is regular then $|X| \geq \sup_{x \in X} 2^{\log \psi(x, X)}$?

We recall that, for a given infinite cardinal m , $\log m = \min\{\tau \mid m \leq 2^\tau\}$. We note that all of the results of this paper are trivial in case CH is assumed. Also, both of the questions are answered affirmatively in case GCH is assumed (since in Question 1, for example, one will even have $|X| \geq 2^{\psi(X)}$).

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