HARMONIC MAPS AND A PINCHING THEOREM FOR POSITIVELY CURVED HYPERSURFACES

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ABSTRACT. In this paper, we establish a theorem of Liouville type for stable harmonic maps in sufficiently pinched, positively curved hypersurfaces of a space form with nonnegative constant curvature. Similar results for the Euclidean sphere S^n have been proved by Y. L. Xin and P. F. Leung, respectively.

1. Statement of result. As is well known, for the Euclidean $n(\geq 3)$ -sphere S^n , there are no nonconstant stable harmonic maps either from S^n to any Riemannian manifold [6] or from any compact manifold to S^n [3]. Its generalizations to submanifolds have been attempted by many authors [3, 4, 5], where the additional conditions imposed on submanifolds are extrinsic. It is natural to find some condition intrinsic for submanifolds. In this paper, we prove the following theorem.

THEOREM. *Let $\tilde{M}^{n+1}(c)$ be an $(n+1)$ -dimensional simply connected space form with constant sectional curvature $c \geq 0$. Suppose that M^n ($n \geq 3$) is a compact hypersurface in $\tilde{M}^{n+1}(c)$ of which the sectional curvature Riem^M satisfies the following pinching condition:*

$$(1) \quad c + a^2 / [(n-2)c + (n-1)a] \leq \text{Riem}^M < c + a$$

for some constant $a > 0$. Then there is no nonconstant stable harmonic map between M^n and any compact Riemannian manifold N , i.e., either from M^n to N or from N to M^n .

COROLLARY 1. *If M^n ($n \geq 3$) is a compact hypersurface in the Euclidean space E^{n+1} with pinching condition: $a/(n-1) \leq \text{Riem}^M < a$ for some constant $a > 0$, then there is no nonconstant stable harmonic map between M^n and any compact Riemannian manifold.*

COROLLARY 2. *If M^n ($n > 4$) is a compact hypersurface in E^{n+1} with $\frac{1}{4} \leq \text{Riem}^M < 1$, then the same conclusion as in Corollary 1 holds.*

REMARK. The condition (1) is sharp in some sense; see an example below.

2. Preliminaries. Let M and N be Riemannian manifolds of dimensions n and m , respectively, and $\phi: M \rightarrow N$ a smooth map. Throughout this paper, we agree on the following ranges of indices unless otherwise stated:

$$1 \leq i, j, k, \dots \leq n; \quad 1 \leq \alpha, \beta, \gamma, \dots \leq m.$$

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We choose local fields of orthonormal frames $\{e_i\}$ and $\{e'_\alpha\}$ in M and N , respectively, and let $\{\omega_i\}$ and $\{\omega'_\alpha\}$ be the fields of dual frames. Under the map ϕ , we have $\phi^*\omega'_\alpha = \sum_i a_{\alpha i}\omega_i$. The energy of ϕ is defined by

$$E(\phi) = \frac{1}{2} \int_M \sum_{\alpha,i} (a_{\alpha i})^2 * 1,$$

where $*1 = \omega_1 \wedge \cdots \wedge \omega_n$. The map ϕ is *harmonic* iff its tension field $\tau = \sum_{\alpha,i} a_{\alpha i} e'_\alpha$ vanishes identically, where $a_{\alpha ij}$ is the covariant derivative of $a_{\alpha i}$ [1].

For any deformation vector field along ϕ , $V = \sum_\alpha v_\alpha e'_\alpha$, the second variation of $E(\phi)$ is [2, 4]

$$(2) \quad E''(\phi) = - \int_M \left[\sum_\alpha v_\alpha \left(\Delta_M v_\alpha + \sum_{i,\beta,\gamma,\delta} v_\delta a_{\beta i} a_{\gamma i} R'_{\beta\alpha\gamma\delta} \right) \right] * 1,$$

where $R'_{\alpha\beta\gamma\delta}$ is the curvature tensor of N and Δ_M is the Laplacian on M . If $E''(\phi) \geq 0$ for any V along ϕ , then the harmonic map ϕ is said to be *stable*.

Now let $\tilde{M}^{n+1}(c)$ be an $(n+1)$ -dimensional simply connected space form with constant sectional curvature $c \geq 0$, and M^n a hypersurface in $\tilde{M}^{n+1}(c)$. We choose a local field of orthonormal frames $e_1, \dots, e_{n+1}$ in $\tilde{M}^{n+1}(c)$ such that, restricted to M^n , the vector e_{n+1} is normal to M^n . Let $\omega_1, \dots, \omega_{n+1}$ be the field of dual frames relative to the frame field chosen above. Then, restricted to M , we have [1]

$$\omega_{n+1} = 0, \quad \omega_{n+1,i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

The Gauss equation of M^n in $\tilde{M}^{n+1}(c)$ is

$$(3) \quad R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk},$$

where R_{ijkl} is the curvature tensor of M^n .

By [4 and 5], we have the following lemma, whose proof is omitted here.

LEMMA 1. *Let M^n ($n \geq 3$) be a compact hypersurface in $\tilde{M}^{n+1}(c)$ with $c \geq 0$. If the second fundamental tensor h_{ij} of M^n satisfies pointwisely*

$$(4) \quad \sum_k (2h_{ik}h_{jk} - h_{kk}h_{ij}) < (n-2)c\delta_{ij}$$

for an orthonormal basis $\{e_i\}$ on M^n , then there is no nonconstant stable harmonic map between M^n and any compact Riemannian manifold.

3. The proof of the Theorem. Let P be an arbitrary point in M^n and $\{\lambda_i\}$ eigenvalues of the matrix (h_{ij}) . It follows from (3) that

$$(5) \quad R_{ijij} = c + \lambda_i \lambda_j \quad (i \neq j)$$

at $P \in M$. The pinching condition (1) together with (5) implies that all of $\{\lambda_i\}$ are nonzero and have the same sign. So, without loss of generality, we may assume that at P

$$(6) \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

and set

$$(7) \quad H = \sum_i \lambda_i.$$

Thus, (4) becomes

$$2\lambda_i^2 - H\lambda_i - (n-2)c < 0$$

for all i , which together with (6) yields that

$$\lambda_i \leq \lambda_n < \frac{1}{4}[H + \sqrt{H^2 + 8(n-2)c}].$$

Hence, we have the following lemma.

LEMMA 2. *Under the assumption (6), the condition (4) is equivalent to*

$$(8) \quad \lambda_n < \frac{1}{4}[H + \sqrt{H^2 + 8(n-2)c}]$$

at $P \in M$.

We now suppose that

$$(9) \quad b^2 \leq \lambda_i \lambda_j < B^2 \quad (i \neq j)$$

for some $0 < b < B$ at $P \in M$.

LEMMA 3. *If $\lambda_1 \geq b$ and B, b in (9) satisfy*

$$(10) \quad B^2 - Lb + (n-1)b^2 = 0,$$

where

$$(11) \quad L = \frac{1}{2}[3(n-1)b + \sqrt{(n-1)^2b^2 + 4(n-2)c}],$$

then the inequality (8) holds.

PROOF. Since $\lambda_1 \geq b$, it follows from (9) and (10) that

$$(12) \quad \lambda_n < B^2/\lambda_1 \leq B^2/b = L - (n-1)b.$$

On the other hand, we have from (11)

$$[3L - 4(n-1)b]^2 = L^2 + 8(n-2)c,$$

i.e.,

$$(13) \quad L - (n-1)b = \frac{1}{4}[L + \sqrt{L^2 + 8(n-2)c}].$$

Substituting (13) into (12), we get

$$(14) \quad \lambda_n < \frac{1}{4}[L + \sqrt{L^2 + 8(n-2)c}].$$

If $H \geq L$, (14) implies (8). So we need only consider the case that $H < L$. Putting

$$(15) \quad L - H = K > 0,$$

we see from (6), (7) and (15)

$$\lambda_n = H - \sum_{j \neq n} \lambda_j \leq H - (n-1)b = L - (n-1)b - K,$$

which together with (13) yields

$$(16) \quad \lambda_n \leq \frac{1}{4}[H + \sqrt{L^2 + 8(n-2)c} - 3K].$$

On the other hand, from the fact that $K > 0$ and $c \geq 0$ it is easy to see that

$$[\sqrt{(L-K)^2 + 8(n-2)c} + 3K]^2 > L^2 + 8(n-2)c,$$

i.e.,

$$(17) \quad \sqrt{H^2 - 8(n-2)c} > \sqrt{L^2 + 8(n-2)c} - 3K.$$

Substituting (17) into (16), we complete the proof of Lemma 3.

LEMMA 4. *If $n \geq 3$, and (10) holds, then (8) is true.*

PROOF. By Lemma 3, we need only consider the case that $\lambda_1 < b$. Since $n \geq 3$, there exist λ_1 and λ_2 ($\leq \lambda_3$) such that, by (9), $\lambda_1 \lambda_2 \geq b^2$.

Construct

$$(18) \quad \lambda'_1 = \lambda'_2 = \frac{1}{2}(\lambda_1 + \lambda_2).$$

Then,

$$(\lambda'_1)^2 = \frac{1}{4}(\lambda_1 + \lambda_2)^2 \geq \lambda_1 \lambda_2 \geq b^2,$$

so that $\lambda'_1 \geq b$. Since (18) implies that $\lambda'_1 = \lambda'_2 \leq \lambda_3$ and $\lambda'_1 + \lambda'_2 + \sum_{k=3}^n \lambda_k = H$, then we can apply Lemma 3 to the case where $0 < \lambda'_1 = \lambda'_2 \leq \lambda_3 \leq \dots \leq \lambda_n$ and conclude that (8) is true. Thus, Lemma 4 is proved completely.

Now, we can easily give the *proof of the Theorem*.

It is clear, by (5), that (1) becomes

$$(19) \quad a^2 / [(n-2)c + (n-1)a] \leq \lambda_i \lambda_j < a \quad (i \neq j)$$

at $P \in M$.

Taking $b = a / \sqrt{(n-2)c + (n-1)a}$, we have

$$(20) \quad a = \frac{1}{2}b[(n-1)b + \sqrt{(n-1)^2b^2 + 4(n-2)c}].$$

Combining (19) with (20), we obtain (9) and (10) at P . By Lemmas 2 and 3, we conclude that (4) holds at P . Since $P \in M$ is arbitrary, the Theorem follows immediately from Lemma 1.

EXAMPLE. In order to employ Lemma 1, the inequality $\text{Riem}^M < c + a$ in (1) cannot be replaced by $\text{Riem}^M \leq c + a$. In such a sense, our pinching condition (4) is sharp. In fact, we may set

$$(21) \quad \lambda_1 = \dots = \lambda_{n-1} = b, \quad \lambda_n = \frac{1}{2}[(n-1)b + \sqrt{(n-1)^2b^2 + 4(n-2)c}],$$

where

$$b = a / \sqrt{(n-2)c + (n-1)a}$$

for some constant $a > 0$. In this case, we have (6) and

$$(22) \quad b^2 \leq \lambda_i \lambda_j \leq \frac{1}{2}b[(n-1)b + \sqrt{(n-1)^2b^2 + 4(n-2)c}] \quad (i \neq j)$$

which together with (5) implies that

$$c + a^2 / [(n-2)c + (n-1)a] \leq \text{Riem}^M \leq c + a.$$

On the other hand, from (21) and (11) we can see that

$$H = (n-1)b + \lambda_n = \frac{1}{2}[3(n-1)b + \sqrt{(n-1)^2b^2 + 4(n-2)c}] = L,$$

which together with (13) yields that

$$\lambda_n = \frac{1}{4}[H + \sqrt{H^2 + 8(n-2)c}],$$

i.e., (8) is not true. Thus, Lemma 1 cannot be applied to this case.

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