PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 2, February 1987

# ADDITIVE GROUPS OF $T$-RINGS

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ABSTRACT. We build on a result of Bowshell and Schultz to give a complete characterization of the groups which occur as the additive groups of $T$-rings. This answers a question of Feigelstock.

In this paper, all groups are abelian. We write $R^+$ for the additive group of a ring $R$, $t(G)$ for the torsion subgroup of a group $G$, and $G_p$ for the $p$-primary component of $G$.

Fuchs [**F1**, Problem 45] posed the problem of classifying those rings $R$ with the property that $R \approx \mathrm{End}_{\mathbf{Z}}(R^+)$. Bowshell and Schultz [**BS**] found one class of rings with this property that they called $T$-rings. A unital ring $R$ is a $T$-ring if the map $m\colon R \otimes_{\mathbf{Z}} R \to R$ induced by multiplication $a \otimes b \to ab$ is an isomorphism. [**BS**] characterized $T$-rings in terms of their additive groups. In order to state their result, we recall some basic facts about torsion-free groups. A torsion-free group has rank one if and only if it is isomorphic to a subgroup of the rational numbers. Rank one groups are completely classified by an invariant called type. A rank one group is the additive group of a unital ring if and only if its type is idempotent under a natural product. Readers unfamiliar with this material are referred to [**F2**, §85] for a thorough treatment.

We are now in a position to state the above-mentioned result.

PROPOSITION 1 [**BS**]. *The following are equivalent for a unital ring $R$:*
(A) *$R$ is a $T$-ring,*
(B) (1) *the quotient $R^+/t(R^+)$ is a rank one group of idempotent type and*
(2) *if $R_p^+$ is nonzero, it is cyclic and $R^+/t(R^+)$ is $p$-divisible.*

From this, one can easily see that a torsion group is the additive group of a $T$-ring if and only if it is cyclic; see [**BS**, 1.4]. This led Feigelstock to ask if the conditions in $B$ above are enough to guarantee that a nontorsion group is the additive group of a $T$-ring [**Fg**, Question 4.7.30]. By Proposition 1, it suffices to determine if such a group is the additive group of some unital ring. In fact, these conditions are not sufficient and we now determine the minor additional restriction needed to give a $T$-ring.

PROPOSITION 2. *Let $G$ be an abelian group which satisfies the conditions in* (B) *above. Then either $G \approx t(G) \oplus G/t(G)$ or else $\bigoplus G_p \leq G \leq \prod G_p$.*

PROOF. Since we assume that each nonzero $p$-component $G_p$ is cyclic, the $G_p$ are pure and bounded, hence they are summands [**F2**, 27.5]. Let $\pi_p\colon G \to G_p$ be a projection map and let $\alpha = \prod \pi_p\colon G \to \prod G_p$ be the product of these maps. Suppose that there is even one element $x$ with infinitely many of the $\pi_p x \neq 0$. In

Received by the editors September 30, 1985 and, in revised form, December 27, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20K99.

this case, we claim that $\alpha$ is an injection. Let $q\colon G \to G/t(G)$ be the quotient map, $q(x) \neq 0$. Take any $g \in G$. If $g \in t(G)$, then of course $\alpha(g) \neq 0$. Next suppose $g \notin t(G)$. Since $G/t(G)$ is assumed to be a rank one group, $q(g)$ is a rational multiple of $q(x)$, i.e. $mq(x) = nq(g)$ for some integers $m$ and $n$. This implies that $mx = ng + t$ for some $t \in t(G)$. Since $x$ has infinitely many nonzero projections $\pi_p x$, so does $mx$. Since $t$ has only finitely many nonzero projections, $ng$ must have infinitely many and so $g$ does also. Thus, $\alpha g \neq 0$ and $\alpha$ is an injection.

Next suppose that there is no element with infinitely many projections. Then every element $g \in G$ can be written $g = k + t$, where $k \in \ker\alpha$ and $t \in t(G)$. Since $\ker\alpha \cap t(G) = 0$, $G = \ker\alpha \oplus t(G)$.

We can now complete the picture of the additive groups of $T$-rings.

PROPOSITION 3. *A group $G$ is the additive group of a $T$-ring if and only if*
(1) *it satisfies the conditions of* (B) *in Proposition* 1 *and*
(2) *there is some $g \in G$ such that for every prime $p$, $\pi_p g$ generates $G_p$.*

PROOF. Say $G \approx R^+$ for a $T$-ring $R$. Proposition 1 shows that (B) holds. It is easy to see that for $1 \in R$, $\pi_p 1$ generates $G_p$ for every $p$.

Conversely, assume that $G$ satisfies (1) and (2). By Proposition 2, $G \approx t(G) \oplus G/t(G)$ or $\bigoplus G_p \leq G \leq \prod G_p$. Suppose that $G \approx t(G) \oplus X$, where $X$ is a rank one, torsion-free group of idempotent type. Write the $g$ given in condition (2) as $g = t + x$, $t \in (G)$, $x \in X$. Since $\pi_p(x) = 0$ for all $p$ and $\pi_p(t) \neq 0$ for only finitely many $p$, $\pi_p(g)$ is nonzero for only finitely many $p$. Since $\pi_p(g)$ generates each $G_p$, only finitely many $G_p$ are nonzero. Since each $G_p$ is cyclic, $t(G)$ is cyclic and carries a unital ring structure. Since $X$ is rank one of idempotent type, it also has unital ring structure. Clearly, $G \approx t(G) \oplus X$ carries the product ring structure.

Suppose $\bigoplus G_p \leq G \leq \prod G_p$. Let $q\colon G \to G/t(G)$ be the quotient map. Choose $g \in G$ such that for all $p$, $\pi_p g$ generates $G_p$ and so that $q(g)$ has an idempotent height sequence. Give each $G_p$ the ring structure that makes $\pi_p g$ the unity of $G_p$ and give $\prod G_p$ the product ring structure. We claim that $G$ is a subring. Clearly, if $t \in \bigoplus G_p$ and $x$ is any element of $\prod G_p$, then $xt \in \bigoplus G_p \leq G$, so we must show that the product of nontorsion elements of $G$ is again in $G$. Take $x, y \in G$ and write $jx = mg + t$, $ky = ng + s$ with $j, k, m, n \in \mathbf{Z}$, $t, s \in t(G)$. Since $q(g)$ has idempotent type and is divisible by $j$ and $k$, it is divisible by $jk$. Choose $z \in G$ with $jkz = mng + u$, $u \in t(G)$. With this choice, $jk(z - xy) \in t(G)$, so $v = z - xy \in t(G)$ and $xy = z - v \in G$. We see that $G$ is a unital subring of $\prod G_p$ and so is a $T$-ring.

It is fairly easy to see that any group satisfying the conditions of (B) has a ring structure which makes the multiplication map $G \otimes G \to G$ an isomorphism. The extra condition in Proposition 3 is simply to insure that this is a unital ring structure.

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